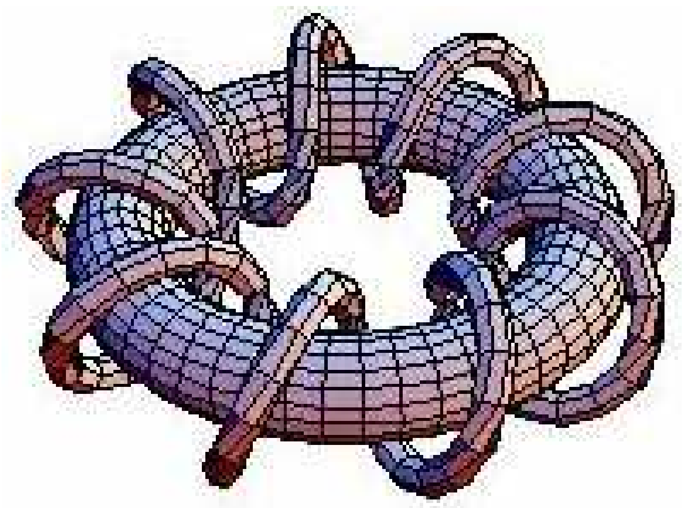
Proceedings

One day international research workshop on

Super-Toroidal Electrodynamics

5 November 2004, University of Southampton

Edited by A.Boardman and N.Zheludev



EPSRC Adventure Fund Project
IOP Quantum Electronics and Photonics Group
EPSRC NanoPhotonics Portfolio Centre, University of Southampton

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List of Participants:

1. Dr. Carlos Vaz, Cambridge University, UK
2. Prof. Alain Dereux, University of Bourgogne, France
3. Dr. Martin McCall, Imperial College, UK
4. Prof. David Hanna, University of Southampton, UK
5. Prof. Arnout Ceulemans, Catholic University of Leuven, Belgium
6. Prof. Will Stewart, University of Southampton, UK
7. Prof. Mikhail Martsenyuk, Perm State University, Russia
8. Dr. Adrian Potts, University of Southampton, UK
9. Dr. Konstantin Vyatkin, Perm State University, Russia
10. Dr. Eugeny Tkalya, Moscow State University, Russia
11. Dr. Gerasim Galitonov, University of Southampton, UK
12. Prof. Hans Schmid, University of Geneva, Switzerland
13. Mr. Justin Llandro, Cambridge University, UK
14. Dr. Vassili Fedotov, University of Southampton, UK
15. Dr. Georgii Afanasiev, Dubna Joint Institute for Nuclear Research, Russia
16. Dr. Kiril Marinov, Salford University, UK
17. Dr. Thomas Moore, Cambridge University, UK
18. Dr. John Chad, University of Southampton, UK
19. Prof. Nikolay Zheludev, University of Southampton, UK
20. Mr. Tom Hayward, Cambridge University, UK
21. Mr. Alexandre Mary, University of Bourgogne, France
22. Prof. Allan Boardman, Salford University, UK (not present on the photo)
23. Mr. Sam Birtwell, University of Southampton, UK (not present on the photo)
24. Dr. Stephan Steinmuller, Cambridge University, UK (not present on the photo)
25. Dr. Kevin MacDonald, University of Southampton, UK (not present on the photo)
26. Mr. Desmond Tse, Cambridge University, UK (not present on the photo)

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One day international research workshop on

Super-Toroidal Electrodynamics

5 November 2004, University of Southampton

Workshop Timetable

Duration of talks: 30 min + 10 min for discussion

9:00 Participants arrive, coffee

9:15 Welcoming message from IOP and EPSRC Adventure Fund Project by **Allan Boardman** and **Nikolay Zheludev**

Morning Session. Chair: Professor **Nikolay Zheludev**, University of Southampton

9:30 *"Electromagnetic toroidal sources and their applications"* by **Georgii Afanasiev**, Joint Institute for Nuclear Research, Dubna, Russia

10:10 *"Topological Invariants in Molecular networks, and their current-like observables"* by **Aurnout Ceulemans**, Catholic University of Leuven, Belgium

10:50 *Coffee break*

11:20 *"Toroidal moments and atomic emission in condensed media"* by **Eugene Tkalya**, Nuclear Research Institute, Moscow state University, Russia

12:00 *Lunch*

Afternoon Session. Chair: Professor **Will Stewart**, University of Southampton

13:30 *"Toroidal moments in spin-ordered crystals"* by **Hans Schmid**, University of Geneva

14:10 *"Toroidal electrodynamics and solid state physics"* by **Mikhail Martsenyuk**, Perm State University, Russia

14:50 *Tea break*

15:20 *"From the detection of the magnetic component of optical near-fields to nano-torus plasmons carrying a magnetic dipole moment at optical frequencies"* by **Alain Dereux**, University of Bourgogne, France.

Open discussion. Chair: Professor **Allan Boardman**, Salford University

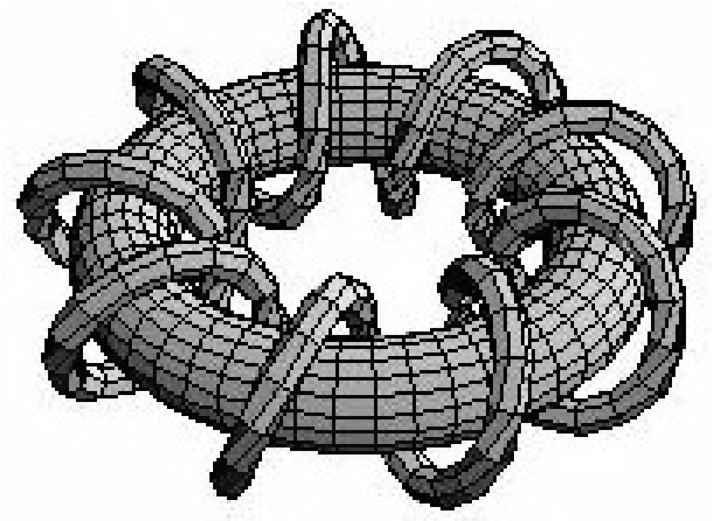
16:00 *Discussion topic: "What could be the crucial experiments in the field of super-toroidal electrodynamics?"*

17:30 Workshop closes

THE DOUGHNUT GAME

Intriguing Propertied of Super-Toroidal Currents

(a brief literature review)



by N.Zheludev

University of Southampton

THE DOUGHNUT GAME

Intriguing Propertied of Super-Toroidal Currents

(a brief literature review)

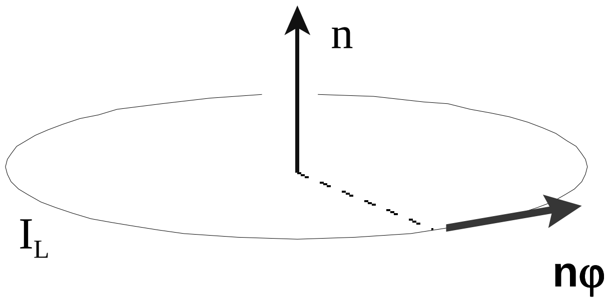
1. From A Current Loop to Supertoroidal Currents
2. Electromagnetic Radiation of Supertoroidal Current
3. Supertoroidal Currents in External Fields
4. Interaction Between Supertoroidal Currents
5. Molecular Toroidal Moments
6. Vector Potential of Supertoroidal Currents and the Aharonov-bohm Effect
7. The “Flying Doughnuts”

Credits to:

Afanasiev, 1990, 1993, 1994, 2001
Afanasiev & Stepanovski, 1995
Afanasiev & Dubovik, 1998
Aharonov & Bohm, 1959
Ceulemans, Chibotrau, Fowles, 1998
Dubovik, Martsenyuk, Saha, 2000
Dubovik & Tugushev, 1990
Feinberg, 1963
Heald, 1988
Helwarth, Nouchi, 1996
Jaklevich & other, 1964
Martsenyuk & Martsenyuk, 1991
Miller, 1984
Nevesski, 1993
Page, 1971
Popov, Kadomtseva & other, 1999
Tolstoi & Spartakov, 1990
Tonomura, 1995
Vlad, 1995
Zeldovich, 1957
Zheludev Ivan, 1976

A CURRENT LOOP

The current loop is equivalent to the magnetization of a sheet enclosed by the loop



Current

$$\boldsymbol{J} = I_L\, \boldsymbol{n}_\varphi\, \delta\,(\rho - \delta)\, \delta\,(z) \;=\; curl\, \boldsymbol{M} \xrightarrow{d \;<<\; R} I_L\;\; \pi\, d^2\; curl\,(\boldsymbol{n}\, \delta^{\,3}(r)\,)$$

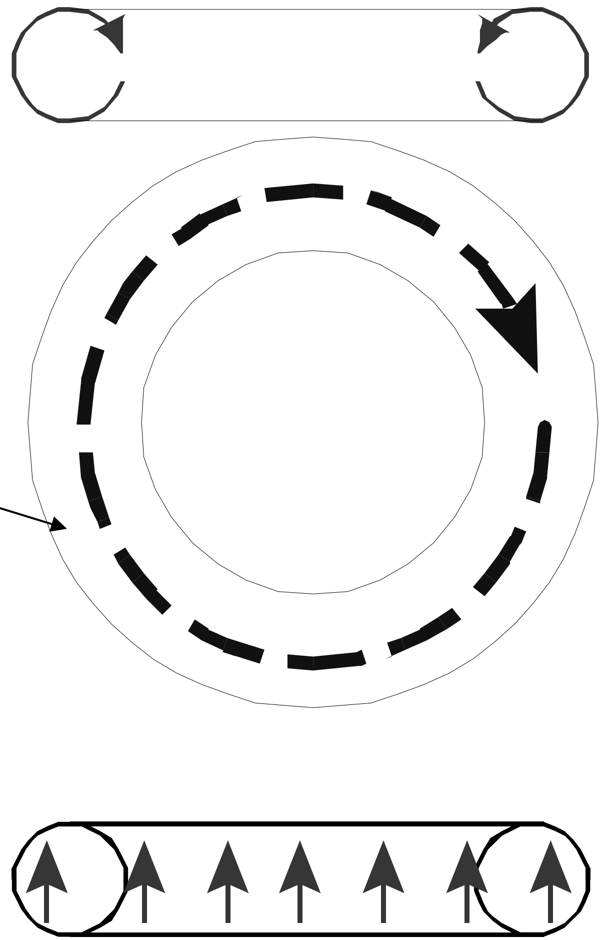
Magnetization

Ampere Law

$$\boldsymbol{M} = I_L\, \boldsymbol{n}\, \theta\,(\rho - \delta)\, \delta\,(z) \xrightarrow{d \;<<\; R} I_L\;\; \pi\, d^2\, \boldsymbol{n}\, \delta^{\,3}(r)$$

POLOIDAL CURRENT ON A TORUS

The poloidal current on a torus is equivalent to the magnetization confined to the interior of the torus



J

Current

M

magnetization

T

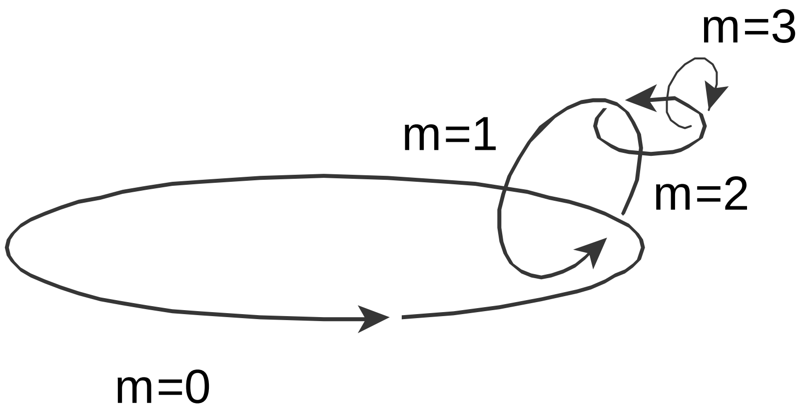
toroidization

$\mathbf{J} = \mathrm{curl}\, \mathbf{M}$

$\mathbf{M} = \mathrm{curl}\, \mathbf{T}$

(since div $\mathbf{M} = 0$)

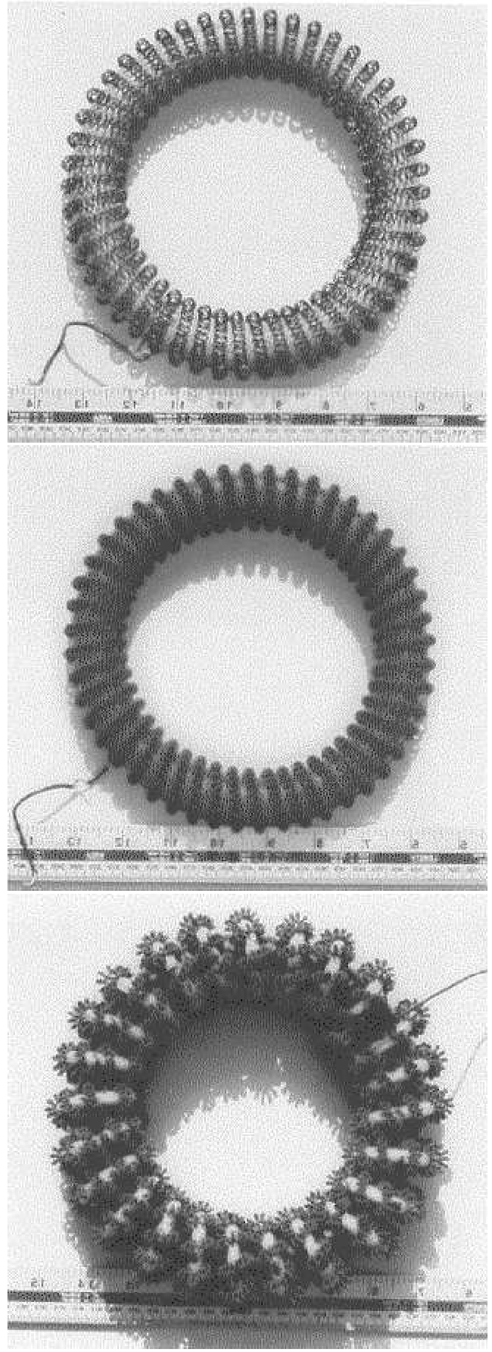
HIERARHY OF TOROIDAL CURRENTS



m=2

m=3

m=4



Current loop

$$\boldsymbol{J_0} = f_0(t)\, curl\ \ (\boldsymbol{n}\, \delta^3(r))$$

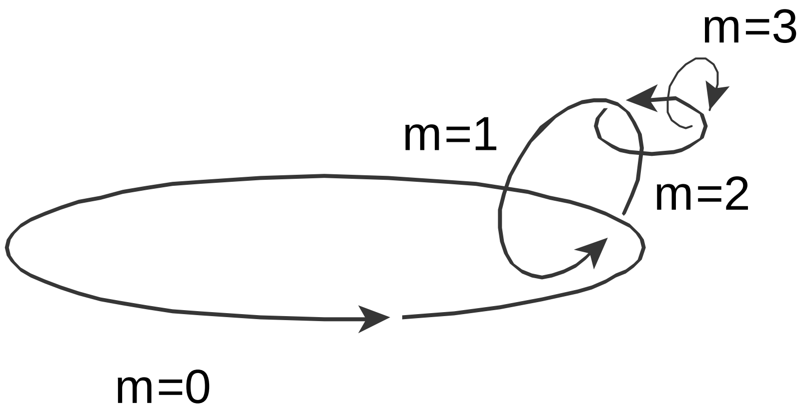
Super-Toroidal current

$$\boldsymbol{J}_m = f_m(t)\, curl^{(m+1)}\, (\boldsymbol{n}\, \delta^3(r))$$

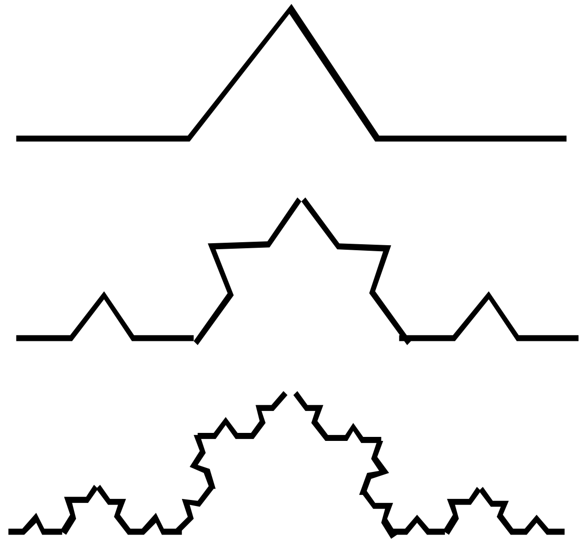
SUPERTOROIDAL CURRENT AS A 3D FRACTAL

Zvirblis, 1995

3D Fractal



2D Fractal



ELECTROMAGNETIC POTENTIALS

Scalar and Vector Potentials

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - grad\,\Phi$$

$$\mathbf{H} = curl\,\boldsymbol{A}$$

Gauge invariance

$$\boldsymbol{A'} = \boldsymbol{A} + \nabla f$$

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial f}{\partial t}$$

ELECTROMAGNETIC FIELDS OF SUPER-TOROIDAL SOLENOIDS

m = 2k, "magnetic" type super-currents

$f_m(t)$ – time dependence of the current

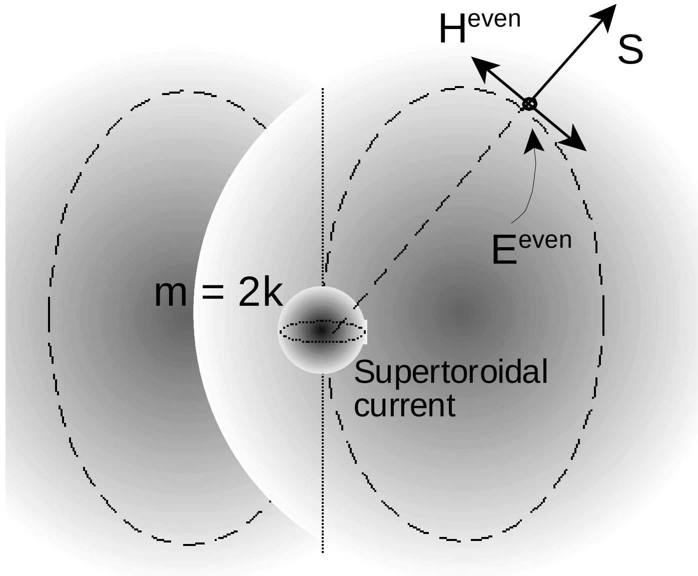
Vector-Potential of radiation

$$\boldsymbol{A}_m \propto \frac{\boldsymbol{r}\times\boldsymbol{n}}{r^2}\,\frac{d^m}{dt^m}D(f_m(t))$$

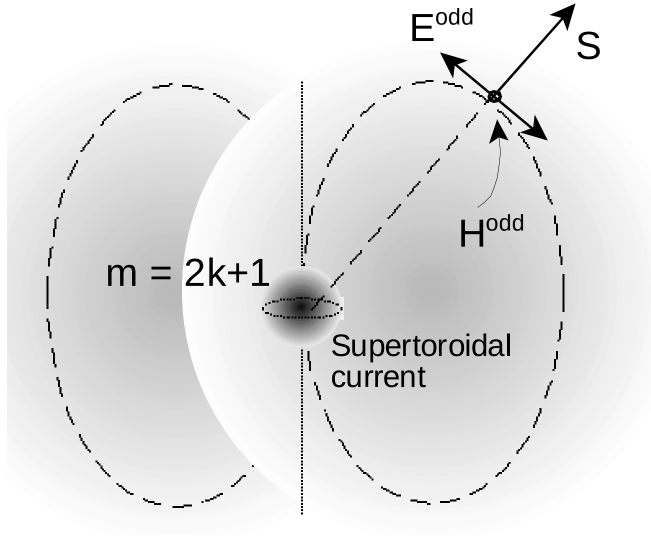
The Pointing vector

$$\boldsymbol{S}_r = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{H} \propto \frac{\sin^2\theta}{r^2}$$

"Magnetic" type super-current



"Electric" type super current



If a current oscillates in:

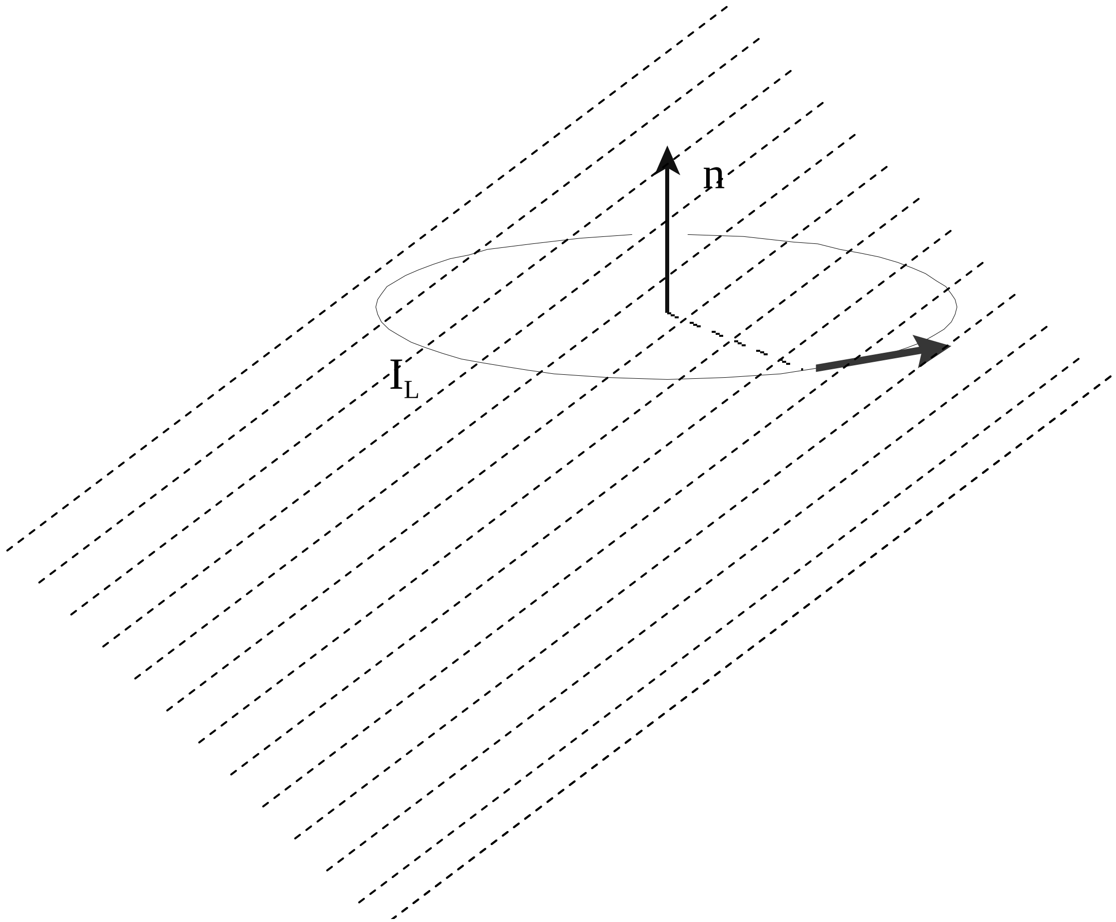
A current loop or super-toroidal coil of even order (m=2k)
= “magnetic” type radiation source

A toroidal coil or supertoroidal coil of odd order (m=2k+1)
= “electric” type (oscillating dipole) radiation source

INTERACTION OF A CURRENT LOOP WITH EXTERNAL FIELS

$$U_{\text{int}\,eraction} = \frac{1}{c}\int \boldsymbol{J}\,\boldsymbol{A}_{ext}\ dV = \frac{1}{c}\int \boldsymbol{M}\,\boldsymbol{H}_{ext}\ dV \xrightarrow{d\to 0} -\mu\,\boldsymbol{H}_{ext}$$

$$\mu = \frac{I_L \pi d^2}{c}\ \boldsymbol{n}$$

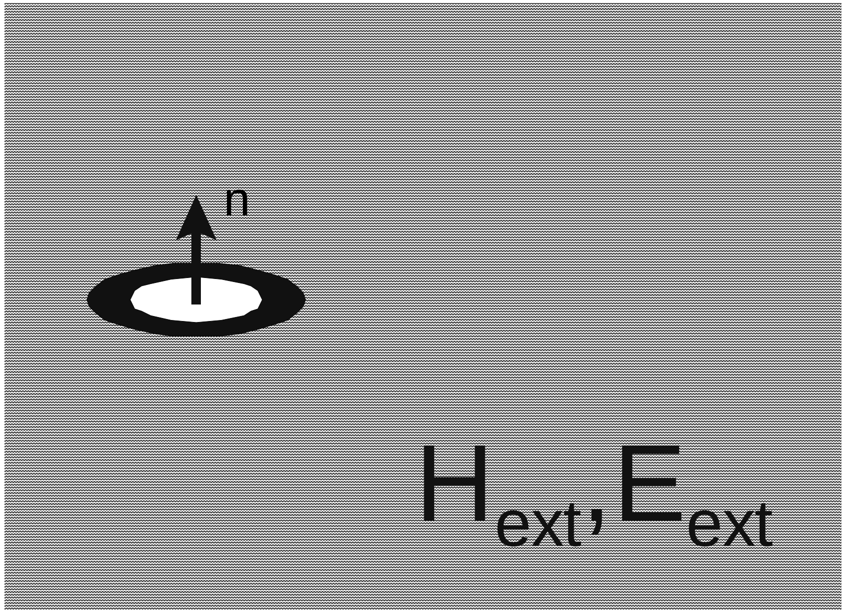


INTEEEERACTION OF SUPER-TOROIDAL CURRENTS WITH EXTERNAL FIELDS

m = 2k, "magnetic" type interaction

f_m (t) – time dependence of the current

$$U_{\text{int}} \propto f_m(t)\,\boldsymbol{n}\,\frac{\partial^m \boldsymbol{H_{ext}}(t)}{\partial t^m}$$



m =o: current loop with H

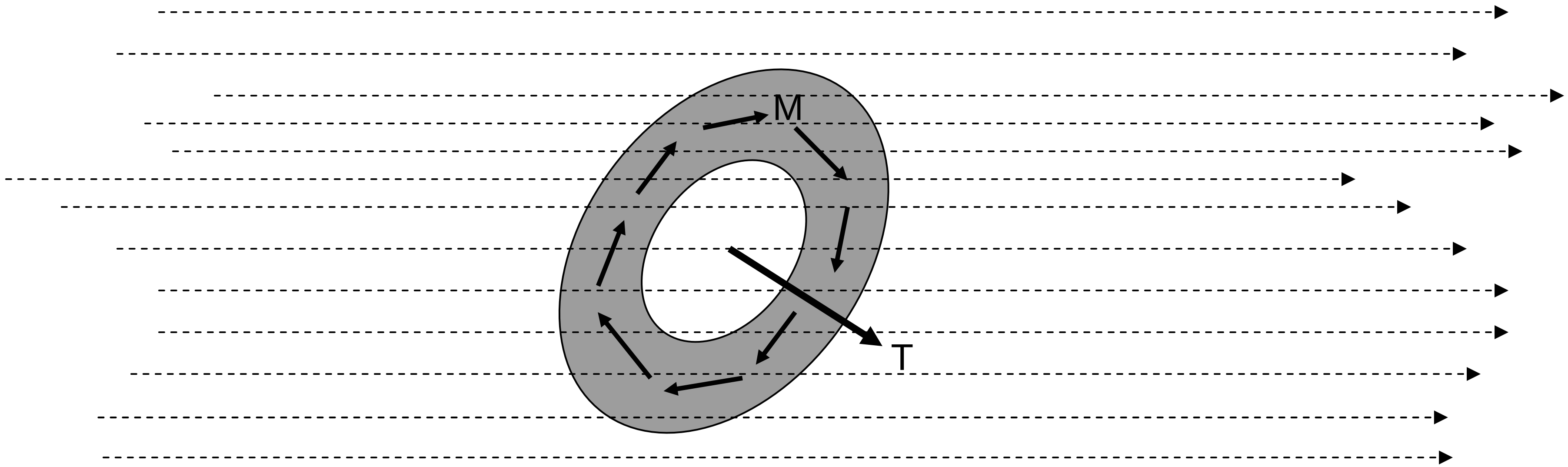
m = 2k+1, "electric" type interactions

$$U_{\text{int}} \propto f_m(t)\,\boldsymbol{n}\,\frac{\partial^m \boldsymbol{E_{ext}}(t)}{\partial t^m}$$

m = 1, toroidal solenoid with current

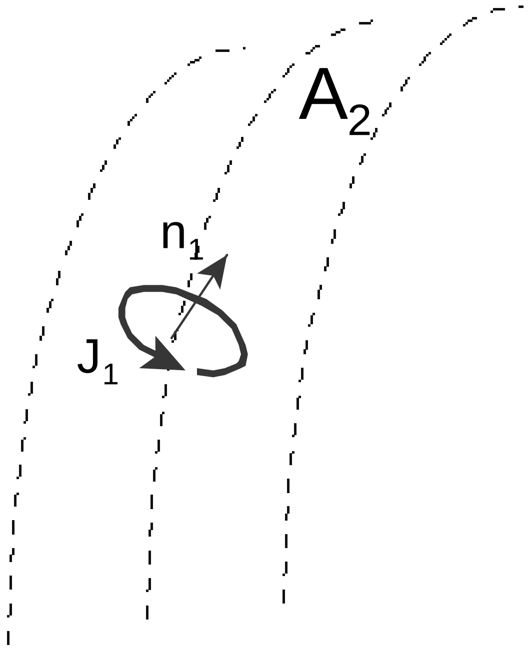
Toroidal Solenoid and Electric Field

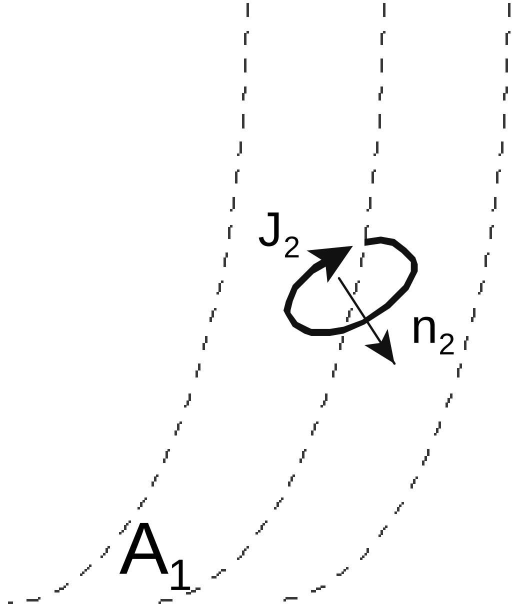
Zeldovich, 1957



Toroidal Solenoid immersed in electrolyte with current
Will be subjected to a moment of force

INTERACTION BETWEEN CURRENTS





$$E_{12} = -\frac{1}{c}\int \boldsymbol{J_1}\,(\boldsymbol{r}-\boldsymbol{r_1})\;\boldsymbol{A_2}\,(\boldsymbol{r}-\boldsymbol{r_2})\,dV$$

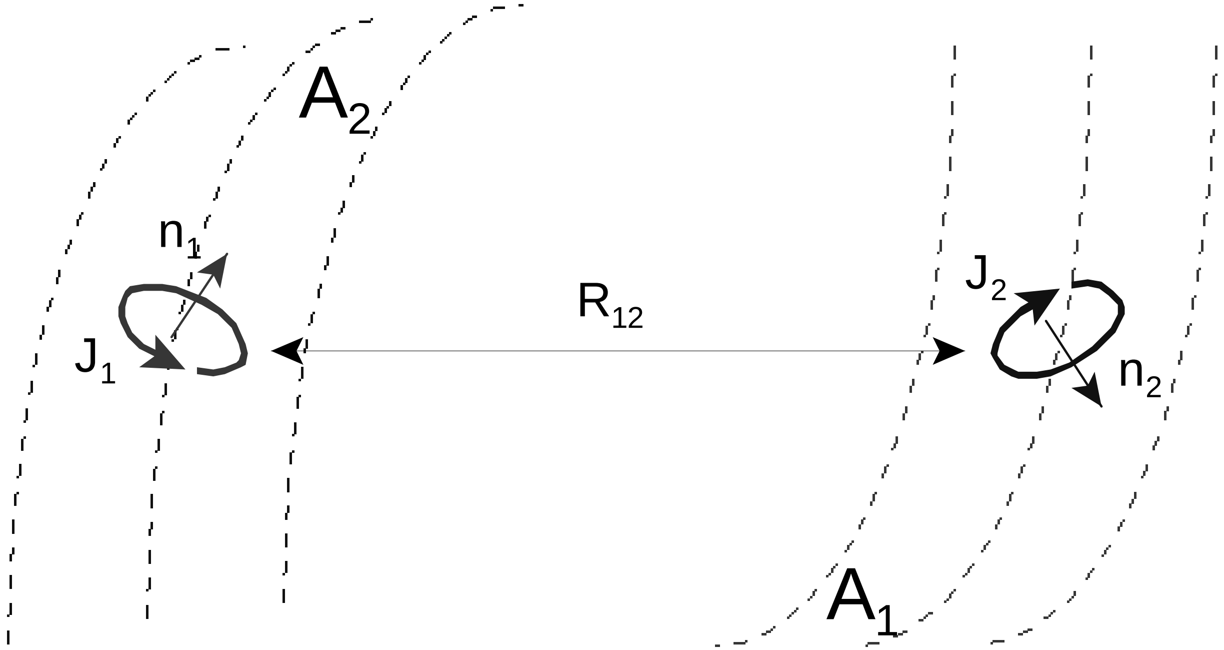
$$E_{21} = -\frac{1}{c}\int \boldsymbol{J_2}\,(\boldsymbol{r}-\boldsymbol{r}_2)\;\boldsymbol{A_1}\,(\boldsymbol{r}-\boldsymbol{r_1})\,dV$$

$$E_{\mathrm{int}} = E_{12} + E_{21}$$

The Reciprocity Relation: $$E_{12} = E_{21}$$

INTERACTION BETWEEN ODD- and EVEN-order TOROIDAL CURRENTS

Afanasiev, 2001



Even type current: $m_1 = 2q$

Odd type current: $m_2 = 2p+1$

$$\boldsymbol{J}_1 = f_1(t)\, curl^{(m_1+1)}\, (\, \boldsymbol{n_1}\ \delta^3\, (\, \boldsymbol{r} - \boldsymbol{r}_1\,))$$

$$\boldsymbol{J_2} = f_2(t)\, curl^{(m_2+1)}\, (\, \boldsymbol{n_2}\ \delta^3\, (\, \boldsymbol{r} - \boldsymbol{r_2}\,))$$

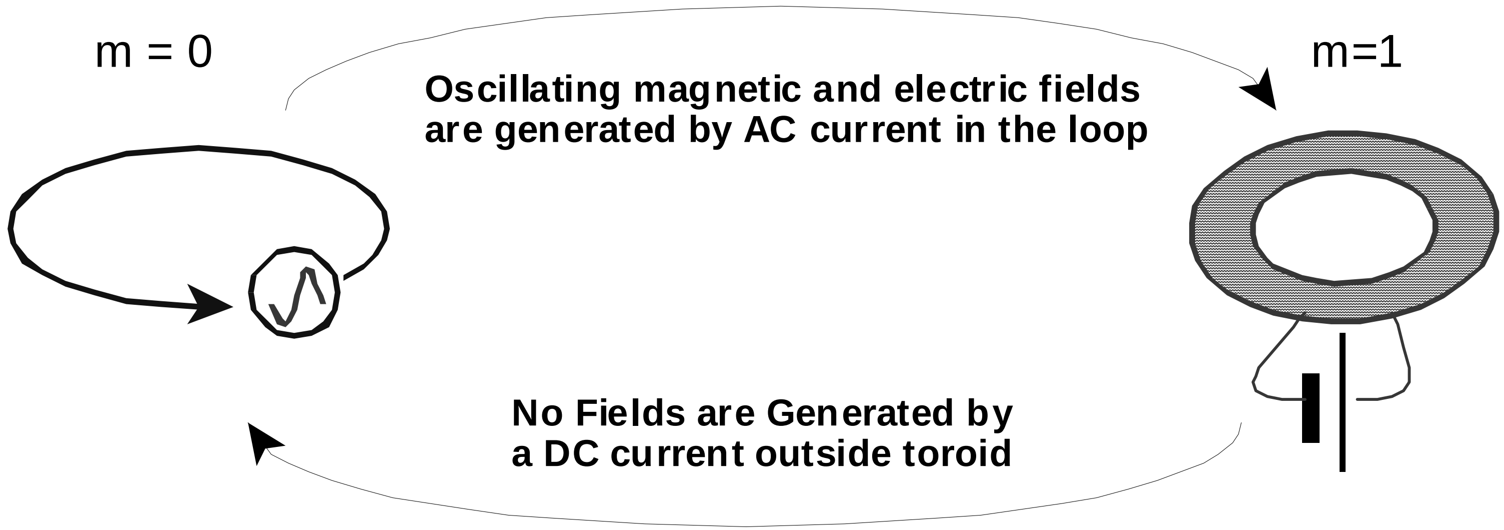
$$E_{12} \propto f_1\,(t) \frac{\boldsymbol{n_1}\ (\boldsymbol{R_{12}} \times \boldsymbol{n_2})}{R^2{}_{12}}\ \frac{\partial^{(m_1+m_2+1)}\ D(\,f_2(t))}{\partial\, t^{\,(m_1+m_2+1)}}$$

$$E_{21} \propto f_2\,(t) \frac{\boldsymbol{n_2}\ (\boldsymbol{R_{21}} \times \boldsymbol{n_1})}{R^2{}_{21}}\ \frac{\partial^{(m_1+m_2+1)}\ D(\,f_1(t))}{\partial\, t^{\,(m_1+m_2+1)}}$$

$$E_{12} \neq E_{21}$$

RECIPROCITY VIOLATION?

VIOLATION OF ACTION-REACTION EQUILITY:
AN EXAMPLE (G.N.Afanasiev, 2001)



$$E_{LT} \propto \frac{\partial^3}{\partial t^3} f_T(t) = 0 \qquad\qquad E_{TL} \propto \frac{\partial^3}{\partial t^3} f_L(t) \neq 0$$

The Reciprocity Relations

Maxwell's equation for current 1

Maxwell's equation for current 2

Some algebra

+ harmonic time dependencies

J_1, J_2~exp(iωt)

$$E_{12} = \int \boldsymbol{J_1}\ \boldsymbol{E_2}\ dV \quad = E_{21} = \int \boldsymbol{J_2}\ \boldsymbol{E_1}\ dV$$

The Lorentz Lemma

$$H_{12} = \int \boldsymbol{J_1}\ \boldsymbol{H_2}\ dV \quad = H_{21} = \int \boldsymbol{J_2}\ \boldsymbol{H_1}\ dV$$

The Feld-Tai Lemma

Limitations of the Reciprocity Relations

Afanasiev, 2001

If time dependent currents are involved the reciprocity relations are applicable if:

1. **The time dependencies are the same for all space points of a particular source**

2. **The time dependencies in source 1 and source 2 are the same**

Free Space Electrodynamics:

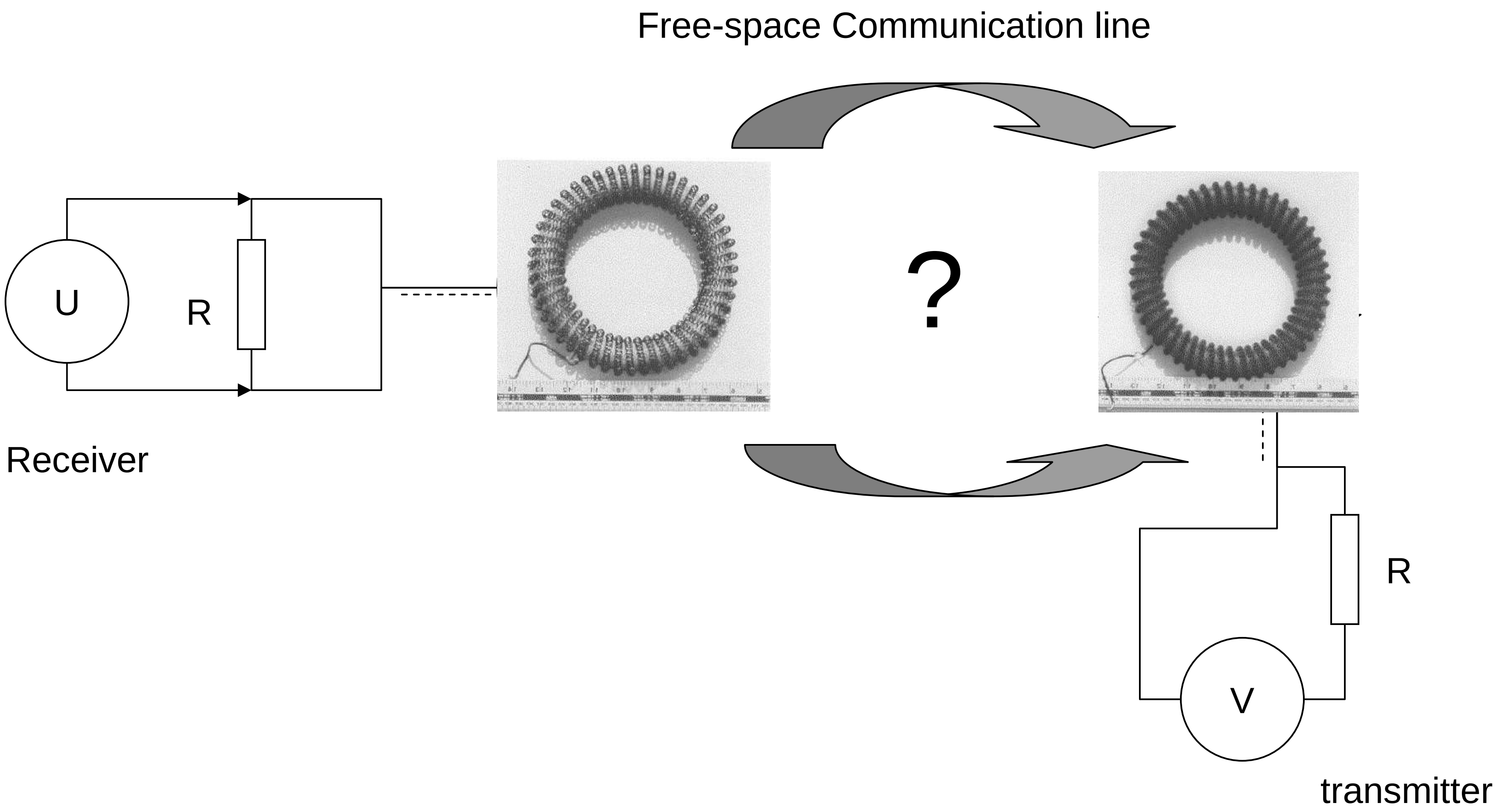
$$E_{12} = \int \boldsymbol{J_1} \ \boldsymbol{E_2} \, dV \ = E_{21} = \int \boldsymbol{J_2} \ \boldsymbol{E_1} \, dV$$

The Lorentz Lemma

The Feld-Tai Lemma

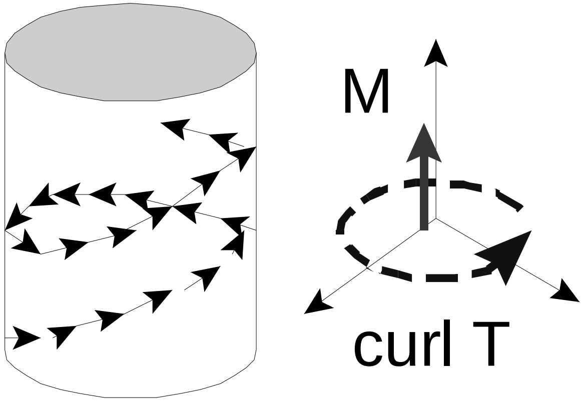
$$H_{12} = \int \boldsymbol{J_1} \ \boldsymbol{H_2} \, dV \ = H_{21} = \int \boldsymbol{J_2} \ \boldsymbol{H_1} \, dV$$

Reciprocity of Radio Transmission



TOROIDAL POLARIZATION IN CONDENCED MATTER PHYSICS

Dubovik, Martsenyuk, Saha, 2000



Magnetization and polarization with account of toroidal moments:

$$\boldsymbol{M} \Rightarrow \boldsymbol{M} + curl\,\boldsymbol{T}^{m}$$

$$\boldsymbol{P} \Rightarrow \boldsymbol{P} + curl\,\boldsymbol{T}^{e}$$

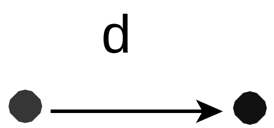
Total Current in Maxwell equations:

$$\boldsymbol{j}_{total} = \boldsymbol{j} + \frac{\partial \boldsymbol{P}}{\partial t} + c\,curl\,\boldsymbol{M} \quad + \frac{\partial\, curl\,\boldsymbol{T}^{e}}{\partial t} + c\,curl\,curl\,\boldsymbol{T}^{m}$$

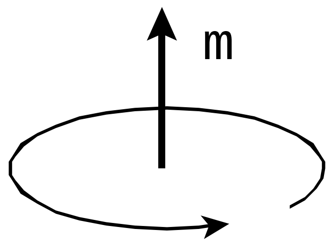
The equation of motion with toroidal force:

$$m\frac{\partial^2 \boldsymbol{r}}{\partial t^2} = e\boldsymbol{E} + \frac{e}{c}[\frac{\partial\,\boldsymbol{r}}{\partial t} \times \boldsymbol{B}\,] \quad + 4\pi e\,(curl\,\boldsymbol{T}^{\boldsymbol{e}} - \frac{1}{c}[\frac{\partial\,\boldsymbol{r}}{\partial t} \times curl\,\boldsymbol{T}^{\boldsymbol{m}}\,])$$

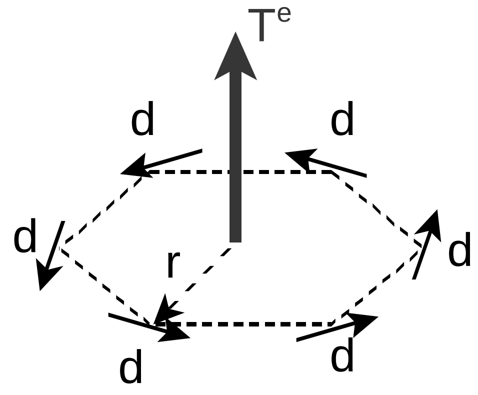
MOLECULAR MOMENTA



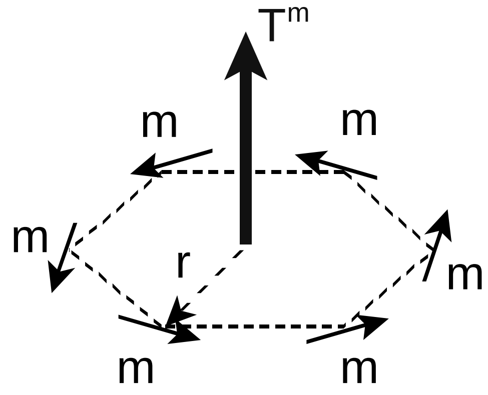
Dipole



Magnetic Dipole



"Electric" toroidal moment



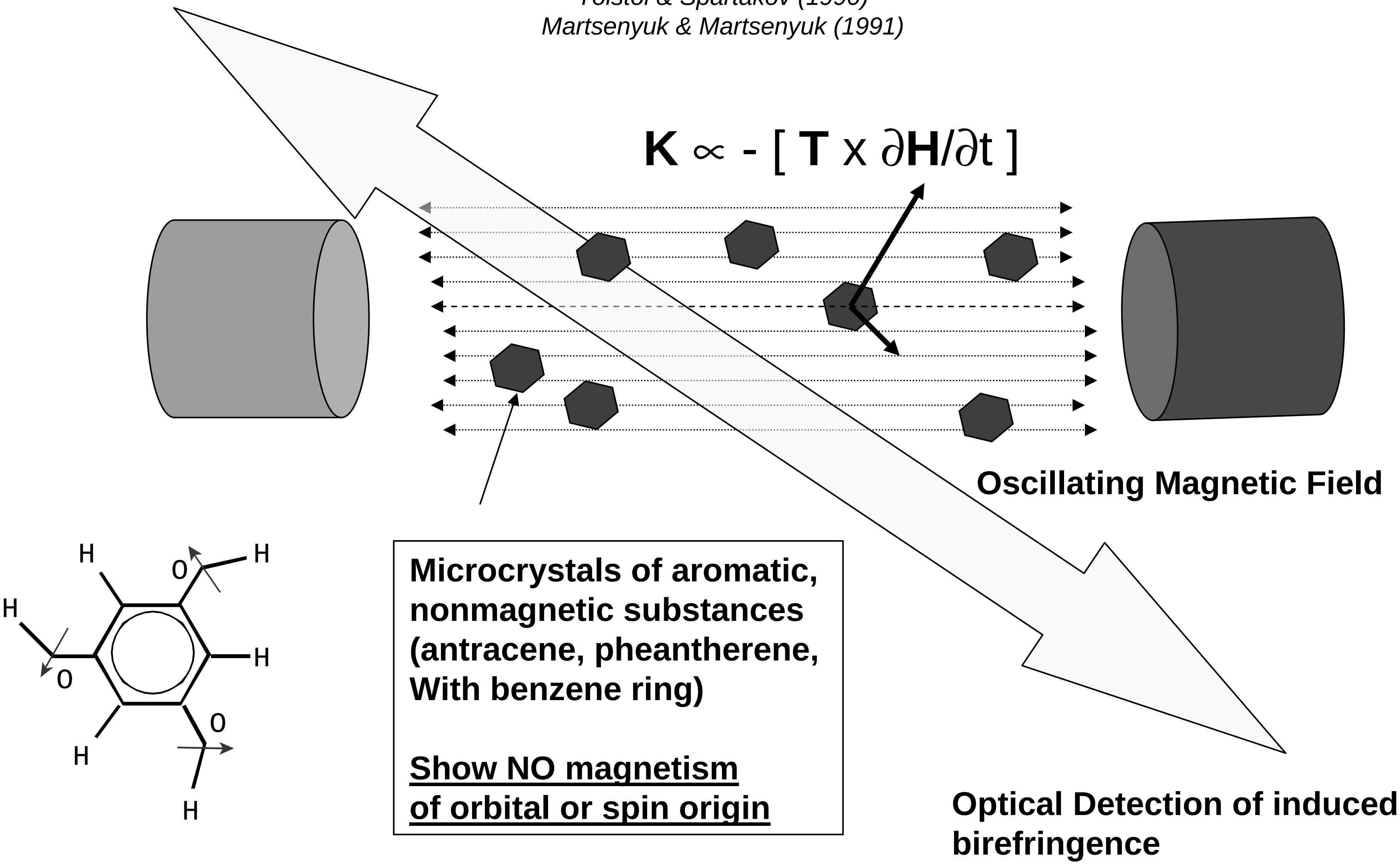
"Magnetic" toroidal moment

$$\boldsymbol{T}^{\boldsymbol{e}} = \frac{1}{2}\sum \boldsymbol{r} \times \boldsymbol{d}$$

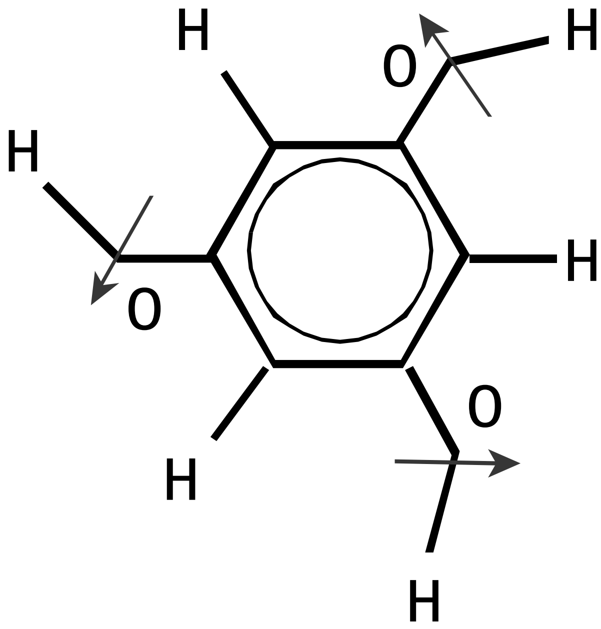
$$\boldsymbol{T}^{\boldsymbol{m}} = \frac{1}{2}\sum \boldsymbol{r} \times \boldsymbol{m}$$

AROMAGNETISM

Tolstoi & Spartakov (1990)
Martsenyuk & Martsenyuk (1991)

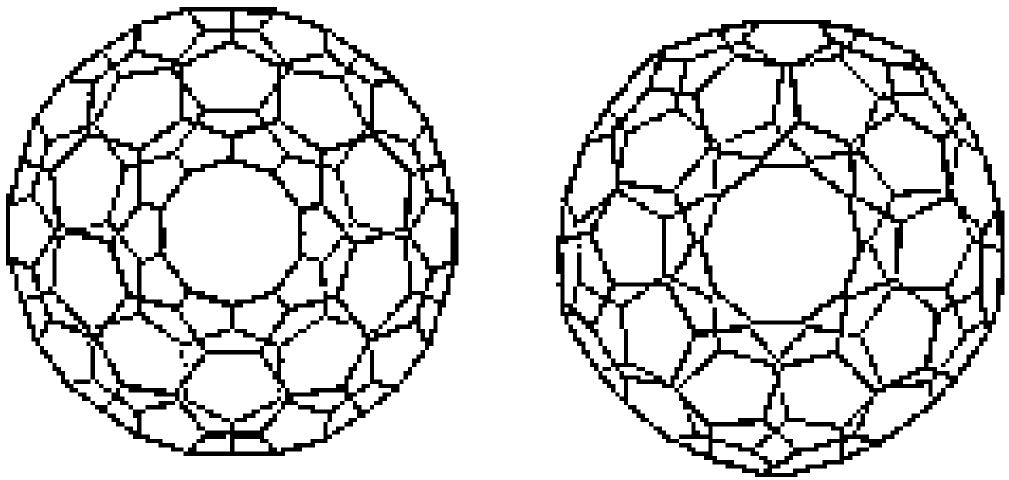


CALCULATION OF MOLECULAR TOROIDAL MOMENTA

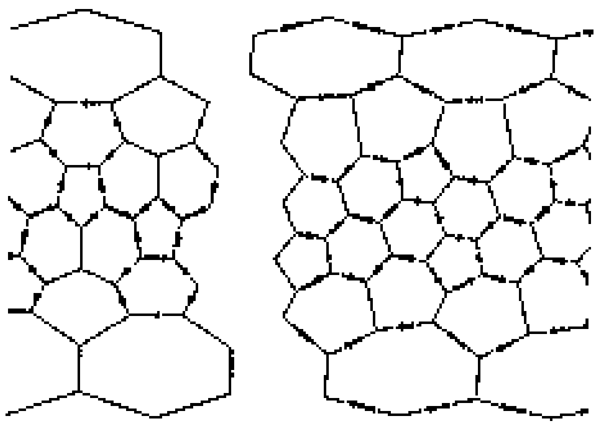


Phloroglycine Molecule shows T^e

Martsenyuk & Martsenyuk, 1991



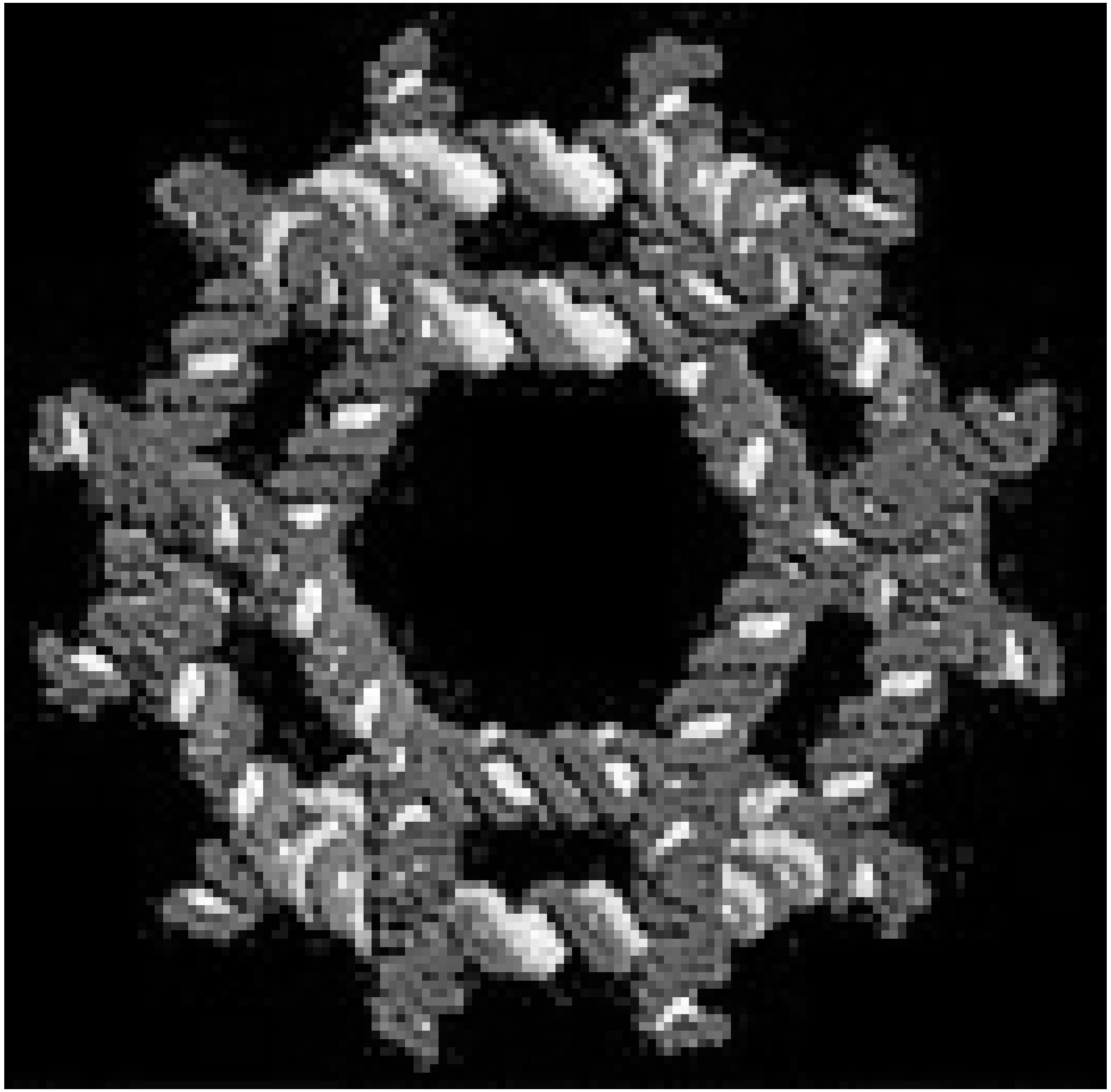
120 atoms Carbon Toroids show T^m



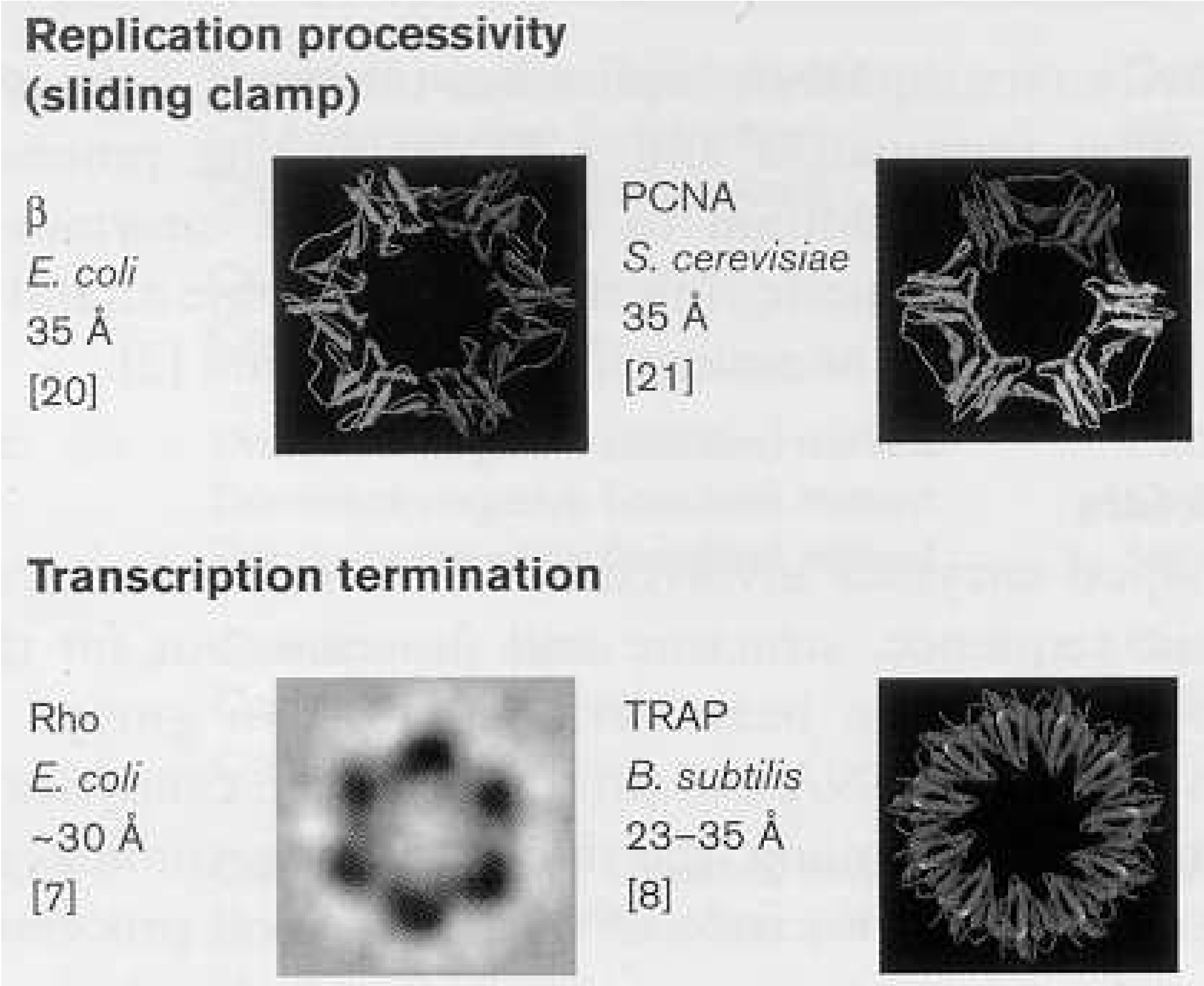
Current included by uniform magnetic field

Ceulemans, Chibotaru & Fowles, 1998

Organic Molecular Tori



Organic Molecular Tori



Organic Molecular Tori

Exonuclease

λ exonuclease

15–30 Å

[2]

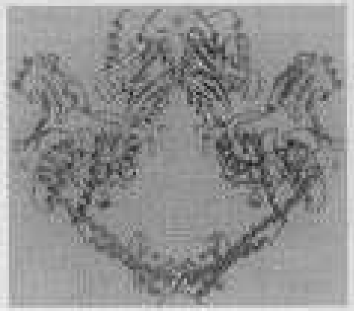
Topoisomerase

Topoisomerase II

S. cerevisiae

25–55 Å

[22]



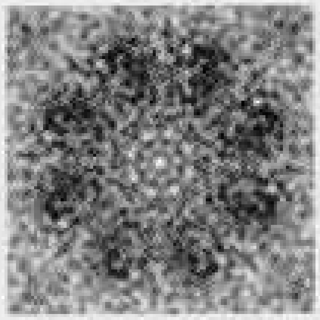
Chromosomal breakpoint recognition

Translin

human

~40 Å

[9]



Phage head–tail connector

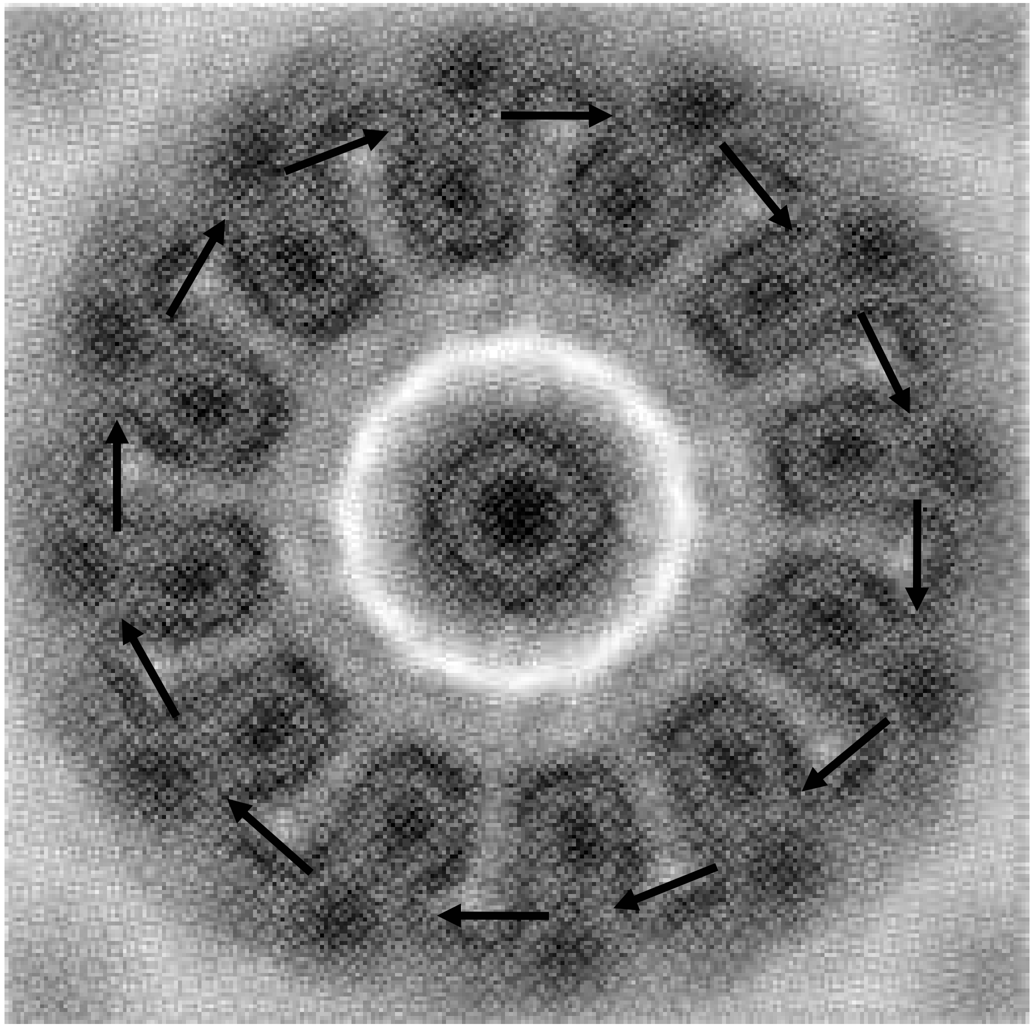
Connector

ϕ29

17–37 Å

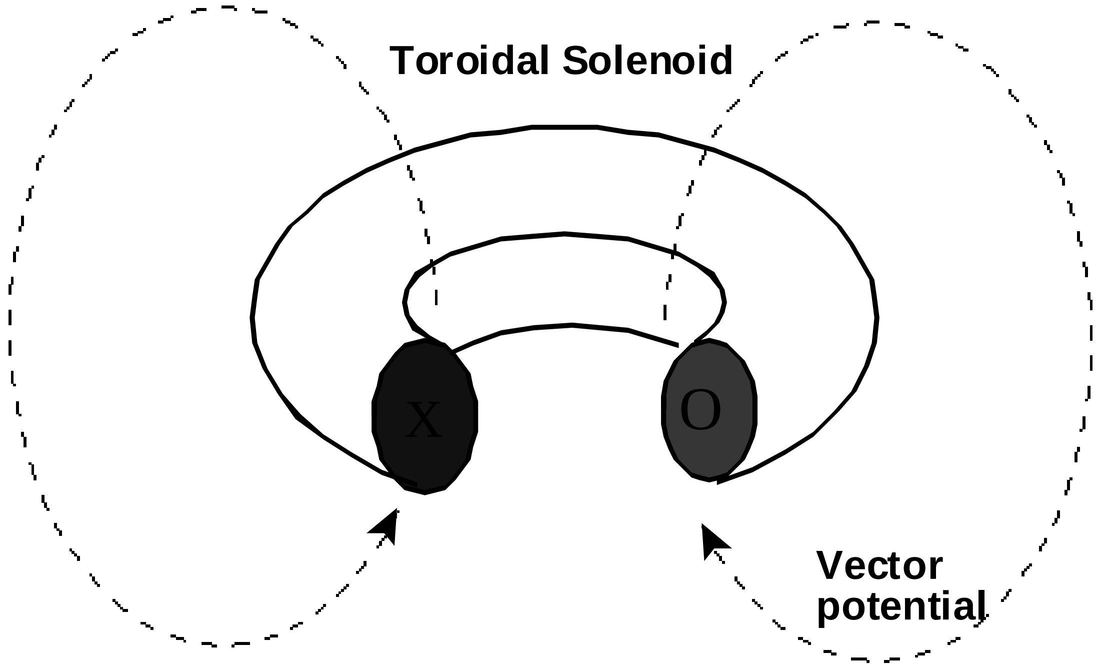
[23]

Molecular toroidal moments?



DC CURRENT IN A TOROIDAL SOLENOID

Page (1971)



DC current:
a Toroidal Solenoid has NO external magnetic & Electric fields
but
HAS a non-vanishing vector-potential field

THE AHARONOV-BOHM EFFECTS AND TOROIDAL CURRENTS

Aharonov-Bohm, 1959

In QM vector potential cannot be illuminated from the basis equations

Do Potential have independent significance from the Fields?

$$\psi_0 \Leftrightarrow H_0$$

$$\psi = \psi_0 e^{i2\pi S/h} \quad \Leftrightarrow \quad H = H_0 + V$$

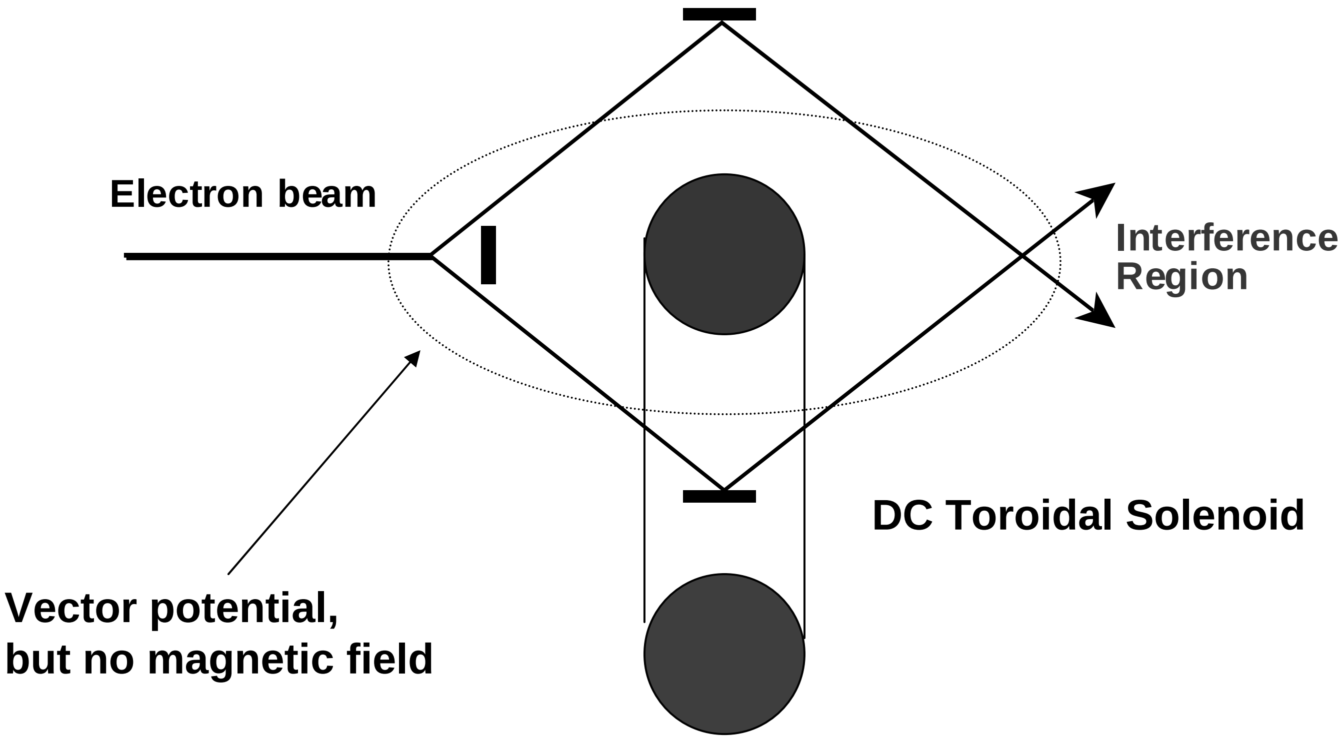
$$where \quad S = \int V(t)dt$$

For a charged particle

$$H = \frac{[\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}]^2}{2m}$$

THE AHARONOV-BOHM EFFECT

Aharonov & Bohm, 1959

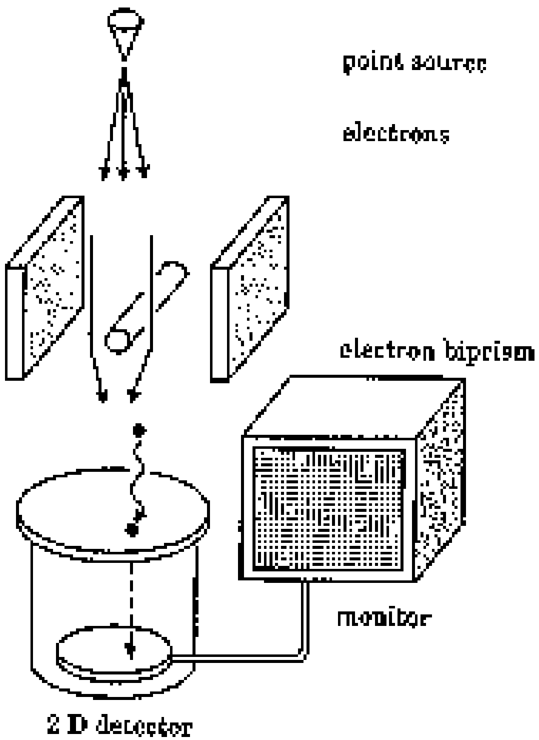


Phase shift depends on the vector potential along the pass (inconceivable in classical electrodynamics, no forces):

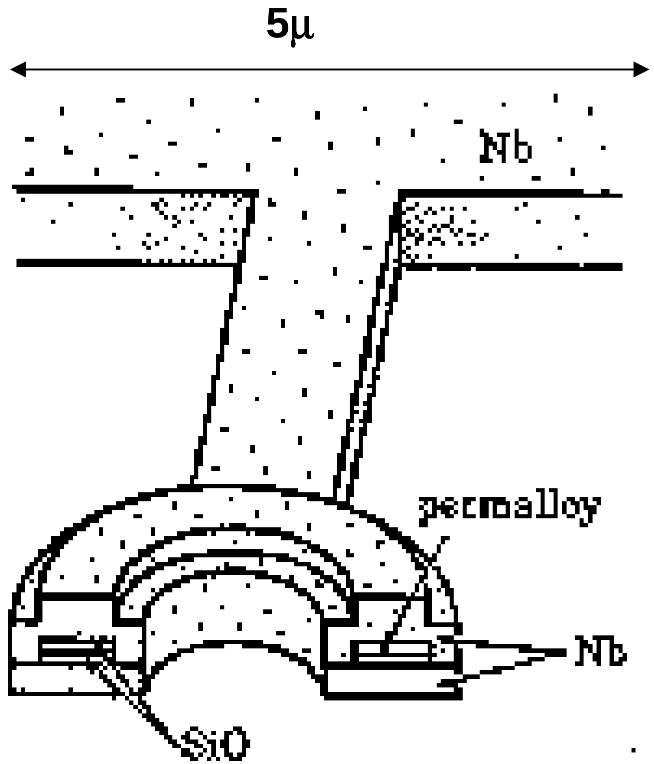
$$\Delta\phi = -\frac{2\pi e}{h}\oint \boldsymbol{A}\, d\boldsymbol{s} = -\frac{2\pi e}{h}\int \boldsymbol{B}\, d\,\boldsymbol{S}$$

TONOMURA's AHARONOV-BOHM EXPERIMENTS

first claim: Jaklevic, Lambe & other, 1964

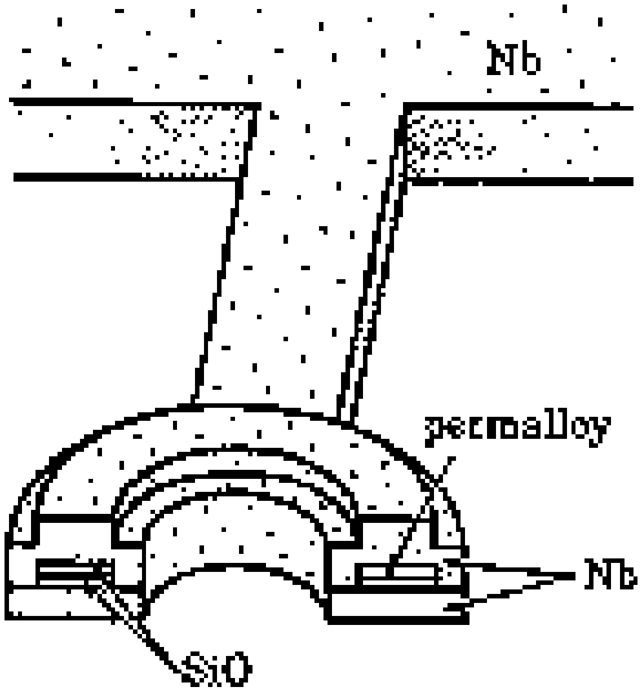


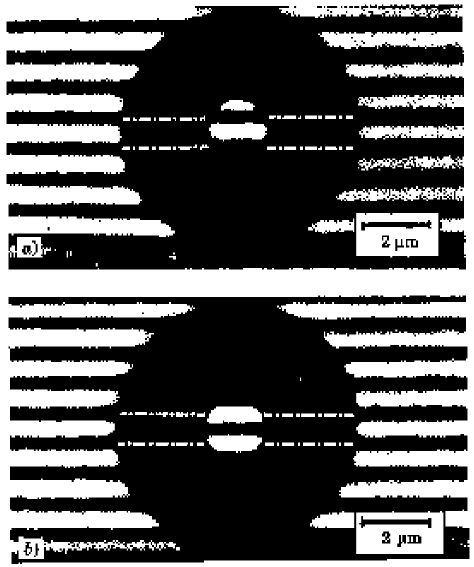
ELECTRON INTERFERENCE EXPERIMENT



PHOTOLITHOGRAPHICALLY MANIFACTURED MAGNETIZED TOROID WITH SUPERCONDUCTION SHEELD (prevents magnetic field to leak) Trapped magnetic flux from 0 to 4h/e

TONOMURA's EXPERIMENTS



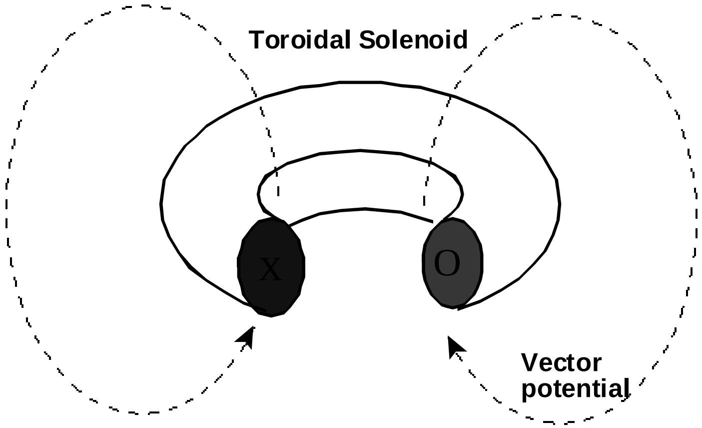


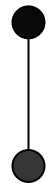
$\varphi=0$
No trapped field

$\varphi=\pi$
Trapped field

NON-RADIATING CONFIGURATION: DC CURRENT IN A TOROIDAL SOLENOID and AN OSCILLATING ELECTRIC DIPOLE

Afanasiev, 2001





Oscillating Dipole

DC Excitation:
a Toroidal Solenoid has NO external magnetic & Electric fields

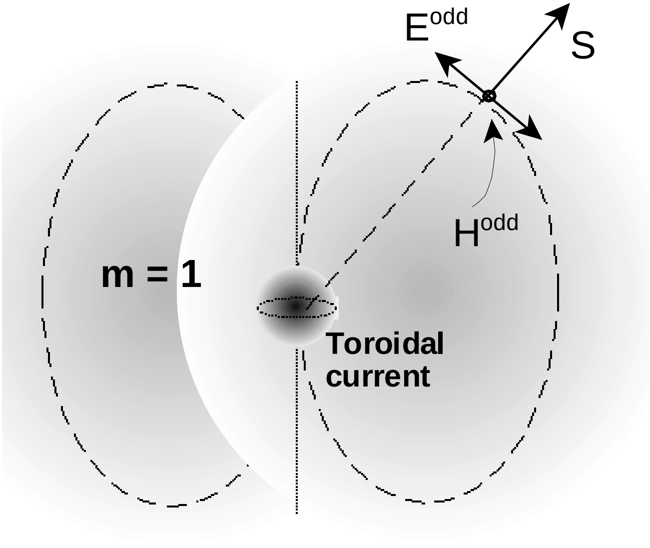
$$\boldsymbol{j}_{dipole} = -\frac{\partial f}{\partial t}\boldsymbol{n}\,\delta^3(\boldsymbol{r}-\boldsymbol{r}_{dipole})$$

Interaction Energy

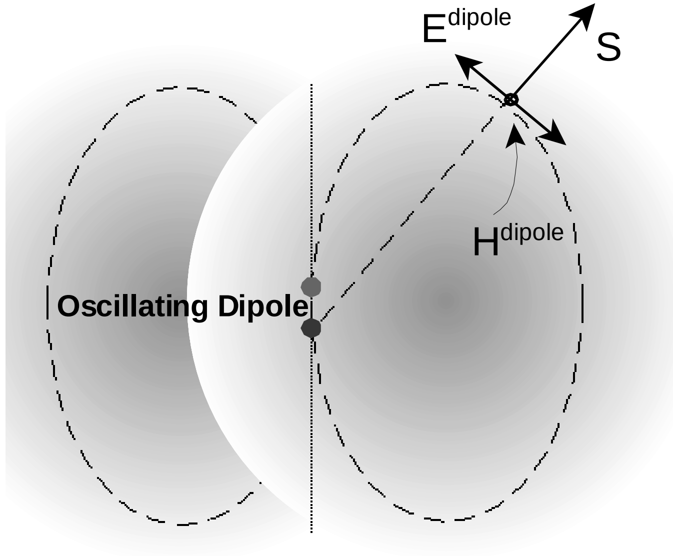
$$U = \int [\rho_{dipole}\Phi_{tor} - \frac{1}{c}\boldsymbol{j}_{dipole}\,\boldsymbol{A}_{tor}]\,dV$$

$$\boldsymbol{U} = -\frac{1}{c}\frac{\partial f}{\partial t}\boldsymbol{n}\,\boldsymbol{A}_{tor} \quad ?!$$

FIELD-LESS SOURCES OF OSCILLATING POTENTIALS??? (Afanasiev 2001)



Oscillating Toroidal Currant



Oscillating Dipole

$$\phi = 0 \quad \Longleftrightarrow \quad \phi = -\frac{(\boldsymbol{n}.\boldsymbol{r})}{cr^2} D(t)$$

$$\boldsymbol{A} = -\frac{\boldsymbol{n}}{c^3 r} G(t) + \frac{\boldsymbol{r}(\boldsymbol{r}.\boldsymbol{n})}{c^3 r^3} F(t) \quad \Longleftrightarrow \quad \boldsymbol{A} = -\frac{\boldsymbol{n}}{c\ r} \frac{\partial}{\partial t} F(t)$$

$$\boldsymbol{E} = -\frac{\boldsymbol{n}}{c^4 r} \frac{\partial}{\partial t} G(t) + \frac{\boldsymbol{r}(\boldsymbol{r}.\boldsymbol{n})}{c^4 r^3} \frac{\partial}{\partial t} F(t) \qquad \boldsymbol{E} = -\frac{\boldsymbol{n}}{c^4 r} \frac{\partial}{\partial t} G(t) + \frac{\boldsymbol{r}(\boldsymbol{r}.\boldsymbol{n})}{c^4 r^3} \frac{\partial}{\partial t} F(t)$$

$$\boldsymbol{H} = \frac{\boldsymbol{r} \times \boldsymbol{n}}{c^4 r^2} \frac{\partial^2}{\partial t^2} D(t) \qquad \boldsymbol{H} = \frac{\boldsymbol{r} \times \boldsymbol{n}}{c^4 r^2} \frac{\partial^2}{\partial t^2} D(t)$$

Electromagnetic Permutations in Free Space

The Helmholtz wave equation

:

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) f(\boldsymbol{r}, t) = 0$$

Solution in the form of perturbations propagating with speed of light :

$$f = f_1(z - ct) + f_2(z + ct)$$

Vector Generalization $\rightarrow$ Polarized Transverse Electromagnetic Harmonic Waves

Flying Doughnuts

Hellwarth & Nouchi, 1996

Ziolkowski (1989) discovered a class of generating Functions which satisfies Helmholtz wave equations:

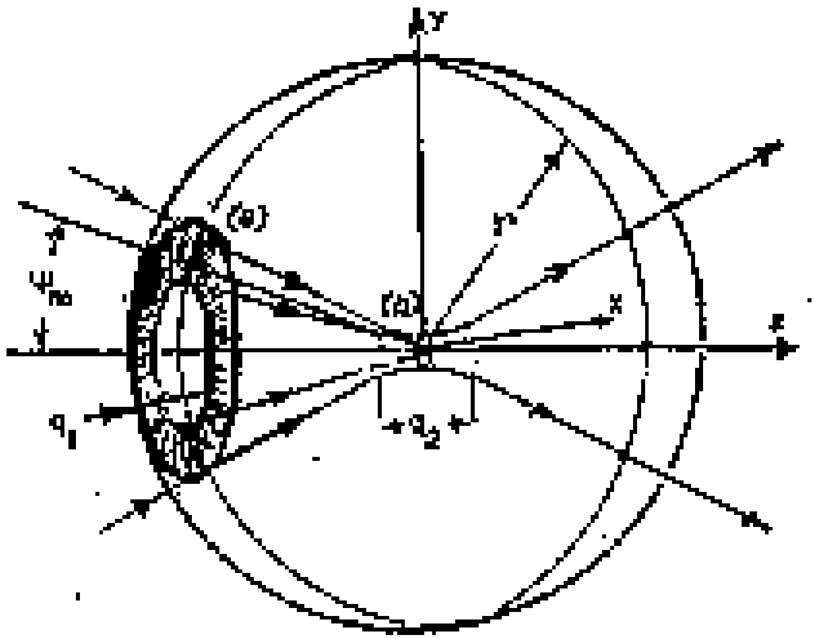
$$f = f_0 \frac{e^{-s/q_3}}{(q_1 + i\tau)(s + q_2)^{\alpha}}$$

Here $q_{1,2\ 3}$ - parameters
S is a function of coordinates and σ

τ = z-ct; σ=z+ct

Vector generalization satisfying Maxwell equations

$$\boldsymbol{A}(\boldsymbol{r},t) = C\,curl\,(\boldsymbol{z}\ f\,(\boldsymbol{r},t))$$

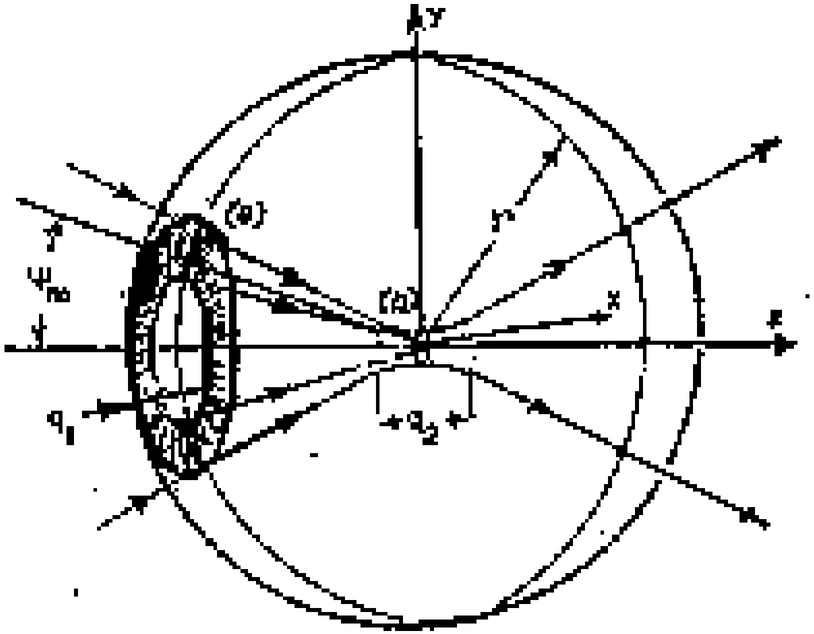


Schematic of ”focused doughnut”

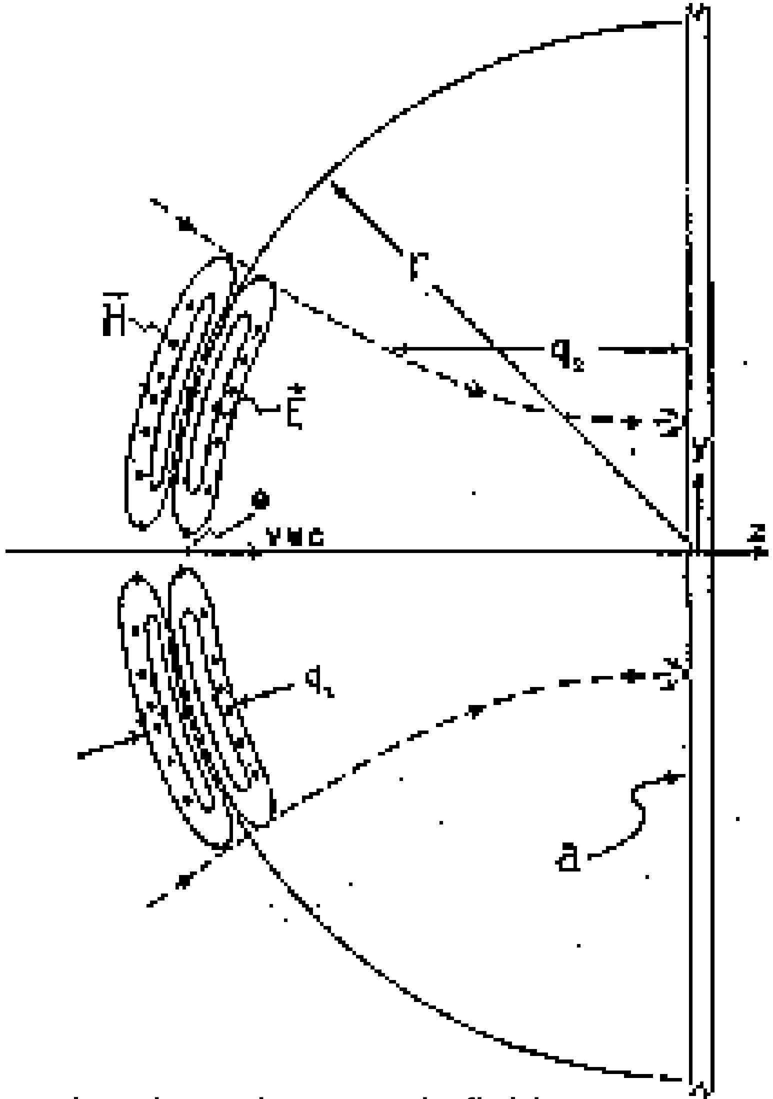
Vector Generalization → “Non-pathological” toroidal wave packets which are neither transverse electric or transverse magnetic

Flying Doughnuts

Hellwarth & Nouchi, 1996

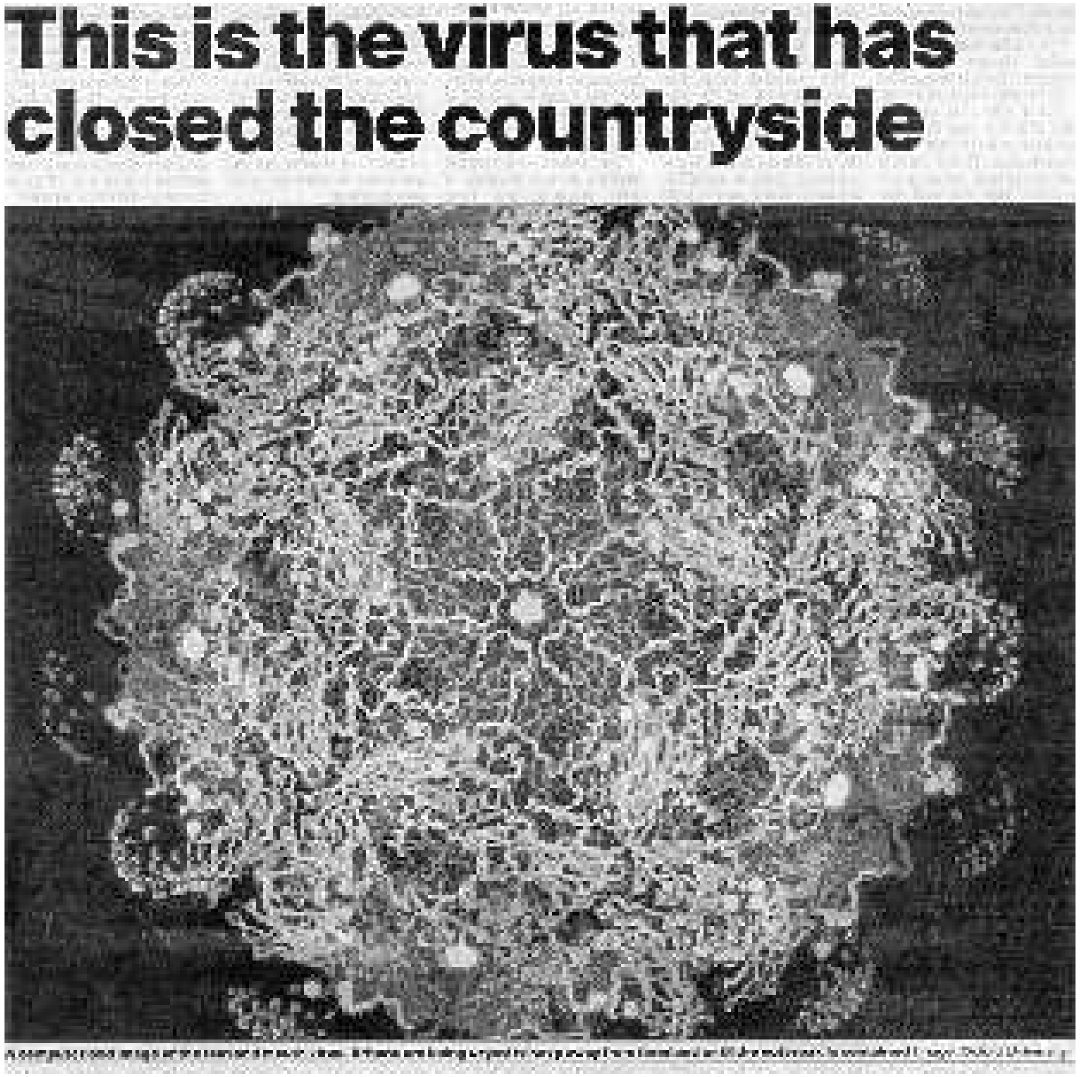


Schematic of ”focused doughnut”



Schematic of the electric and magnetic fields
of a one-cycle electromagnetic “flying doughnut”

THE END



This is the virus that has closed the countryside

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Simplest sources of electromagnetic fields as a tool for testing the reciprocity-like theorems

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Abstract

Electromagnetic fields of the simplest time-dependent sources (current loop, electric dipole, toroidal solenoid, etc), their interactions with an external electromagnetic field and between themselves are found. They are applied to the analysis of the Lorentz and Feld–Tai lemmas (or reciprocity-like theorems) having numerous applications in electrodynamics, optics, radiophysics, electronics, etc. It is demonstrated that these lemmas are valid for more general time dependences of the electromagnetic field sources than it was suggested up to now. It is shown also that the validity of reciprocity-like theorems is intimately related to the equality of electromagnetic action and reaction: both of them are fulfilled or violated under the same conditions. Conditions are stated under which reciprocity-like theorems can be violated. A concrete example of their violation is presented.

1. Introduction

The reciprocity theorem has a long history in physics. It originates from the third Newtonian law stating equality of action and reaction. Later, Rayleigh, in the first volume of his encyclopaedic treatise '*Theory of Sound*' [1] proved certain relations between the forces acting between two physical systems and the displacements induced by them. Since there is no time retardation in the Newtonian mechanics, this statement looks almost trivial. Furthermore, Rayleigh applied the reciprocity theorem to optics [2]. We quote him:

> Suppose that in any direction (i) and at any distance r from a small surface (S) reflecting in any manner there be situated a radiant point (A) of given intensity, and consider the intensity of reflected vibrations at any point B situated in direction ϵ and at distance r' from S. The theorem is to the effect that the intensity is the same as it would be at A if the radiant point were transferred to B.

He gave no proof of this statement referring to the analogy with mechanical systems treated in the '*Theory of Sound*' and to the optical Lambert law. In all probability, Lorentz [3] was the first to have formulated the electric part of reciprocity theorem in its modern form. This theorem has numerous applications in the theory of electric circuits [4], optics [5, 6], electron diffraction [7], radiophysics science [8, 9] and biomedical engineering [10]. The magnetic part of the reciprocity theorem was obtained by Feld [11] and Tai [12] in the same year, 1992. It was rederived by Monzon [13] in 1996 who, without knowing the above papers, pointed out numerous applications of this theorem. Other applications of the Feld–Tai lemma were given by Lakhtakia in his book [14].

The aim of this consideration is to use electromagnetic fields (EMFs) of simplest sources (current loop, toroidal solenoid (TS) and electric dipole) for the study of the reciprocity-like theorems.

The plan of our exposition is as follows. In section 2, we present the formalism of elementary vector potentials (EVPs). Although they are exposed in many textbooks and treatises (see, e.g., [15–17]), the lack of coordination between them is so large that we prefer to give a self-consistent exposition. In section 3, we apply EVPs to the pure current time-dependent sources of EMFs. Special attention is paid to the current loop and TS as well as to their interaction with an external EMF. Various limiting cases of TSs with periodical current are investigated. The EMF of the time-dependent electric dipole and its interaction with an external EMF are studied in section 4. The EMFs of more complicated point-like toroidal sources and their mutual interactions are treated in section 5.

In section 6, by applying the Lorentz and Feld–Tai lemmas to the charge–current sources studied in previous sections, we find that these lemmas are fulfilled under more general conditions than it was known up to now. This obliges us to consider the derivation of the Lorentz and Feld–Tai lemmas more carefully. This is done in section 7, where it is shown that the reciprocity-like theorems are satisfied in the same cases when the equality of action and reaction is fulfilled. New reciprocity-like theorems are obtained in the same section, yet their physical meaning remains unclear to us. The conditions are stated under which the reciprocity-like theorems can be violated and a concrete example demonstrating this violation is presented. A short discussion of the results obtained is given in section 8.

2. Elementary vector potentials

Consider charge $\rho(\vec{r},t)$ and current $\vec{j}(\vec{r},t)$ densities confined to a finite volume V. Let their time dependence be periodical:

$$\rho = \rho_0 \exp(\mathrm{i}\omega t) \qquad \vec{j} = \vec{j}_0 \exp(\mathrm{i}\omega t). \tag{2.1}$$

When presenting ρ and $\vec{j}$ in such a complex form, one should keep in mind the static limit of the problem treated. For example, if one operates with pure current densities and wants to have the time-independent current in a static limit, then one puts

$$\vec{j} = \vec{j}_0 \exp(\mathrm{i}\omega t) \qquad \rho = 0$$

and, after all calculations, takes the real parts of the EMF strengths (see section 3, where the EMF of a current loop and a TS are considered). On the other hand, if one desires to obtain the time-independent charge distribution in a static limit, then one puts

$$\vec{j} = \omega\vec{j}_0 \exp(\mathrm{i}\omega t) \qquad \rho = i\rho_0 \exp(\mathrm{i}\omega t) \qquad \rho_0 = \operatorname{div} \vec{j}_0$$

and, after all calculations, takes the imaginary parts of the EMF strengths (see section 4, where the EMF of an oscillating electric dipole is treated).

The electromagnetic potentials outside space region V, to which the charge–current densities are confined, are given by

$$\Phi(\vec{r},t) = -4\pi \mathrm{i}k \sum h_l(kr) Y_{lm}(\theta,\phi) q_{lm}$$
$$\vec{A}(\vec{r},t) = -\frac{4\pi \mathrm{i}k}{c} \sum \vec{A}_{lm}(\tau,\vec{r}) a_{lm}(\tau) \tag{2.2}$$

where $h_l(kr) \equiv h_l^{(2)}(kr) = j_l(kr) - \mathrm{i}n_l(kr)$ is the spherical Hankel function of the second kind, j_l and n_l are the spherical Bessel and Neumann functions ($j_l = J_{l+1/2}\sqrt{\pi/2x}$, $n_l = N_{l+1/2}\sqrt{\pi/2x}$); $Y_{lm}(\theta,\phi)$ are the usual spherical harmonics; and $\vec{A}_{lm}(\tau,\vec{r})$ are the elementary vector potentials (EVPs). Values for $\tau = E, L$ and M correspond to the electric, longitudinal and magnetic EVPs, respectively. Their manifest forms are given by

$$\vec{A}_{lm}(L) = \frac{1}{k}\vec{\nabla} h_l Y_{lm}$$

$$\vec{A}_{lm}(E) = -\frac{1}{k\sqrt{l(l+1)}} \operatorname{curl}(\vec{r}\times\vec{\nabla}) h_l Y_{lm}$$

$$\vec{A}_{lm}(M) = -\frac{\mathrm{i}}{\sqrt{l(l+1)}} h_l (\vec{r}\times\vec{\nabla}) Y_{lm}. \tag{2.3}$$

If not indicated, the arguments of the spherical Bessel functions (j_l, n_l) will be kr, and $\cos\theta$ will be the argument of the adjoint Legendre polynomials (P_l^m). In what follows, we closely follow the Rose treatise [15] with the exception that instead of his non-standard radial functions, the usual spherical Bessel functions are used. EVPs satisfy the following equations:

$$\operatorname{curl} \vec{A}_{lm}(M) = \mathrm{i}k\vec{A}_{lm}(E) \qquad \operatorname{curl} \vec{A}_{lm}(E) = -\mathrm{i}k\vec{A}_{lm}(M).$$

It is useful to write out the spherical components of EVP in a manifest form

$$[\vec{A}_l^m(E)]_\theta = \frac{1}{\sqrt{l(l+1)}} \frac{(l+1)h_{l-1} - lh_{l+1}}{2l+1} \frac{\mathrm{d}}{\mathrm{d}\theta} Y_{lm}$$

$$[\vec{A}_l^m(E)]_\phi = \frac{m}{\sin\theta\sqrt{l(l+1)}} \frac{(l+1)h_{l-1} - lh_{l+1}}{2l+1} Y_{lm}$$

$$[\vec{A}_l^m(E)]_r = \sqrt{l(l+1)} \frac{1}{kr} h_l Y_{lm}$$

$$[\vec{A}_l^m(M)]_\theta = \frac{\mathrm{i}m}{\sin\theta\sqrt{l(l+1}} h_l Y_{lm}$$

$$[\vec{A}_l^m(M)]_\phi = -\frac{\mathrm{i}h_l}{\sqrt{l(l+1)}} \frac{\partial Y_{lm}}{\partial\theta} \qquad [\vec{A}_l^m(M)]_r = 0. \tag{2.4}$$

The multipole coefficients (or form factors) $a_{lm}(\tau)$ entering into (2.2) are defined as

$$q_{lm} = \int j_l Y_{lm}^* \rho \,\mathrm{d}V$$

$$a_{lm}(L) = -\frac{1}{k}\int j_l Y_{lm}^* \operatorname{div}\vec{j}\,\mathrm{d}V = \frac{1}{k}\int j_l Y_{lm}^* \dot{\rho} = \mathrm{i}cq_{lm}$$

$$a_{lm}(E) = -\frac{1}{k\sqrt{l(l+1)}}\int \operatorname{curl}(\vec{r}\times\vec{\nabla}) j_l Y_{lm}^* \vec{j}\,\mathrm{d}V$$

$$= \frac{k}{\sqrt{l(l+1)}}\int j_l Y_{lm}^* (\vec{r}\vec{j})\,\mathrm{d}V$$

$$+\frac{1}{k\sqrt{l(l+1)}}\int [(l+1)j_l - krj_{l+1}] Y_{lm}^* \dot{\rho}\,\mathrm{d}V$$

$$a_{lm}(M) = -\frac{\mathrm{i}}{\sqrt{l(l+1)}}\int j_l Y_{lm}^* (\vec{r}\operatorname{curl}\vec{j})\,\mathrm{d}V$$

$$= \frac{\mathrm{i}}{\sqrt{l(l+1)}}\int j_l Y_{lm}^* \operatorname{div}(\vec{r}\times\vec{j})\,\mathrm{d}V. \tag{2.5}$$

To escape ambiguities, by $\dot{\rho}$ we mean $-\operatorname{div}\vec{j}$. The EMF strengths are given by

$$\vec{H} = \frac{4\pi k^2}{c}\sum [\vec{A}_{lm}(E)a_{lm}(M) - \vec{A}_{lm}(M)a_{lm}(E)]$$

$$\vec{E} = -\frac{4\pi k^2}{c}\sum [\vec{A}_{lm}(E)a_{lm}(E) + \vec{A}_{lm}(M)a_{lm}(M)]. \tag{2.6}$$

For the axisymmetric charge–current distributions, only the $m = 0$ components survive:

$$q_{lm} = \delta_{mo} q_l \qquad a_{lm}(E) = \delta_{mo} a_l(E)$$

$$a_{lm}(M) = \delta_{mo} a_l(M)$$

$$[\vec{A}_l(E)]_\theta \equiv [\vec{A}_l^0(E)]_\theta = \frac{(l+1)h_{l-1} - lh_{l+1}}{[l(l+1)(2l+1)4\pi]^{1/2}} P_l^1$$

$$[\vec{A}_l(E)]_r \equiv [\vec{A}_l^0(E)]_r = \frac{1}{kr}\left[\frac{l(l+1)(2l+1)}{4\pi}\right]^{1/2} h_l P_l$$

$$[\vec{A}_l(M)]_\phi \equiv [\vec{A}_l^0(M)]_\phi = -\mathrm{i}\left[\frac{2l+1}{4\pi l(l+1)}\right]^{1/2} h_l P_l^1. \tag{2.7}$$

2.1. Historical remarks

Formulae of this section may be found in a number of textbooks. Unfortunately, the differences in notation used are is so large that we prefer to collect here the formulae needed for the subsequent exposition.

3. Pure current densities

When only current densities are present ($\rho = 0$), then $q_{lm} = 0$, $A_{lm}(L) = 0$, and

$$a_{lm}(E) = \frac{k}{\sqrt{l(l+1)}} \int j_l Y_{lm}^*(\vec r \vec j)\,\mathrm{d}V \tag{3.1}$$

while $a_{lm}(M)$ has the same form (2.5). Taking into account the fact that

$$j_l(x) \sim (x)^l/(2l+1)!! \qquad n_l(x) \sim -(2l-1)!!/(x)^{l+1} \quad \text{for} \quad x \to 0$$

one obtains in the static limit ($k \to 0$)

$$a_{lm}(E) \to \frac{k^{l+1}}{\sqrt{l(l+1)}} \frac{1}{(2l+1)!!} \int r^l Y_{lm}^*(\vec r \vec j)\,\mathrm{d}V$$

$$a_{lm}(M) \to \frac{\mathrm{i}}{\sqrt{l(l+1)}} \frac{k^l}{(2l+1)!!} \int r^l Y_{lm}^* \operatorname{div}(\vec r \times \vec j)\,\mathrm{d}V. \tag{3.2}$$

The integrals entering into these equations are usually called electric and magnetic moments, respectively. On the other hand, the toroidal moment corresponding to the current density $\vec j$ was defined in [18] as

$$T_{lm} = -\frac{\sqrt{\pi} l}{c(2l+1)} \times \int r^{l+1} \left[\vec Y_{l,-1,m}^* + \sqrt{\frac{l}{l+1}} \frac{1}{l+3/2} \vec Y_{l,l+1,m}^* \right] \vec j\,\mathrm{d}V \tag{3.3}$$

where $\vec Y_{j,l,m}^*$ are the so-called vector spherical harmonics (see, e.g., [15] for their definition). In view of the identities

$$\begin{aligned} &\int r^{l+1} \left[\vec Y_{l,l-1,m}^* + \sqrt{\frac{l}{l+1}} \frac{1}{l+3/2} \vec Y_{l,l+1,m}^* \right] \vec j\,\mathrm{d}V \\ &= -\sqrt{\frac{2l+1}{l}} \frac{1}{(l+1)(2l+3)} \int \operatorname{curl}(\vec r \times \vec\nabla) r^{l+2} Y_{lm}^* \vec j\,\mathrm{d}V \\ &= -\frac{1}{(l+1)(2l+3)} \sqrt{\frac{2l+1}{l}} \left[(l+3) \int r^{l+2} Y_{lm}^* \operatorname{div} \vec j\,\mathrm{d}V \right. \\ &\left. + 2(2l+3) \int r^l Y_{lm}^* (\vec r \vec j)\,\mathrm{d}V \right] \end{aligned} \tag{3.4}$$

established in [19], one obtains for the pure current densities

$$T_{lm} = \frac{2\sqrt{\pi}}{c(l+1)} \sqrt{\frac{l}{2l+1}} \int r^l Y_{lm}^* (\vec r \vec j)\,\mathrm{d}V. \tag{3.5}$$

Therefore, a toroidal moment T_{lm}, in the absence of charge density ($\rho = 0$), up to a factor independent of the geometric parameters of the current distribution coincides with the electric moment (2.5) of this distribution.

3.1. EMF of a current loop

Let the current loop lie in the $z = 0$ plane with its symmetry axis along the z axis. Then, its current density is given by

$$\vec j_L = I_0 \vec n_\phi \delta(\rho - d)\delta(z). \tag{3.6}$$

Since $\vec r \vec j_L = 0$, only the magnetic form factors differ from zero

$$a_l^m(M) = \delta_{m0} a_l(M)$$

$$a_l(M) = \mathrm{i} I_0 d \left[\frac{\pi(2l+1)}{l(l+1)} \right]^{1/2} j_l(kd) P_l^1(0). \tag{3.7}$$

Here $P_l^m(x)$ is the adjoint Legendre function. Since $P_l^1(0) = 0$ for l even, only odd multipole coefficients contribute to the EMF of the current loop ($P_{2n+1}^1(0) = (-1)^{n+1}(2n+1)!!/2^n n!$). Therefore, for the current loop

$$\vec H = \frac{4\pi k^2}{c} \sum \vec A_l(E) a_l(M)$$

$$\vec E = -\frac{4\pi k^2}{c} \sum \vec A_l(M) a_l(M). \tag{3.8}$$

From the facts that: (i) $\vec r \vec E = 0$ and (ii) $P \vec A_l(E) = (-1)^{l+1} \vec A_l(E)$ it follows [15, 17] that the radiation field of the current loop is of a magnetic type (P is the parity operator).

When the time dependence of ρ and $\vec j$ is $\cos\omega t$, the non-vanishing EMF strengths are given by

$$\begin{aligned} E_\phi &= -\frac{2\pi I_0 d k^2}{c} \times \sum_{l=odd} \frac{2l+1}{l(l+1)} (\cos\omega t\, j_l + \sin\omega t\, n_l) P_l^1 j_l(kd) P_l^1(0) \\ H_\theta &= \frac{2\pi I_0 d k^2}{c} \sum_{l=odd} \frac{1}{l(l+1)} P_l^1 j_l(kd) P_l^1(0) \\ &\quad \times \{\cos\omega t[(l+1)n_{l-1} - l n_{l+1}] \\ &\quad - \sin\omega t[(l+1) j_{l-1} - l j_{l+1}]\} \\ H_r &= \frac{2\pi I_0 k d}{cr} \times \sum_{l=odd} (2l+1)(n_l \cos\omega t - j_l \sin\omega t) P_l j_l(kd) P_l^1(0). \end{aligned} \tag{3.9}$$

To estimate the number of $a_l(M)$ contributing to the sums in (3.9), we need the asymptotic behaviour of $J_\nu(x)$ for x fixed and $\nu \gg 1$. This is given by (see [20], ch 8)

$$J_\nu(x) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{xe}{2\nu} \right)^\nu.$$

It follows from this equation that the number n ($l = 2n+1$) of terms contributing to (3.9) with $a_l(M)$ given by (3.7) should be slightly greater than $0.7kd$.

Consider the particular following cases.

(1) In the static case ($k \to 0$), one obtains

$$j_l(kd) \sim (kd)^l/(2l+1)!! \qquad n_l(kr) \sim -(2l-1)!!/(kr)^{l+1}$$

$$E_\phi = 0 \qquad H_\theta = \frac{2\pi I_0 d}{cr^2} \sum \frac{1}{l+1} \frac{d^l}{r^l} P_l^1 P_l^1(0)$$

$$H_r = -\frac{2\pi I_0 d}{cr^2}\sum \frac{d^l}{r^l} P_l P_l^1(0). \qquad (3.10)$$

The term $l = 1$ of this sum

$$H_\theta = \frac{\pi I_0 d^2}{cr^3}\sin\theta \qquad H_r = \frac{2\pi I_0 d^2}{cr^3}\cos\theta$$

corresponds to the field of a magnetic dipole of the power $m = \pi I_0 d^2/c$ oriented along the z axis.

(2) When the radius d of the loop is so small that $kd \ll 1$, only the $l = 1$ term contributes to (3.9). Then, EMF strengths are equal to

$$E_\phi = \frac{\pi I_0 d^2 k^2}{cr}\sin\theta\left(\cos\psi - \frac{1}{kr}\sin\psi\right)$$

$$H_r = \frac{2\pi I_0 d^2 k\cos\theta}{cr^2}\left(\sin\psi + \frac{1}{kr}\cos\psi\right)$$

$$H_\theta = -\frac{\pi I_0 d^2 k^2\sin\theta}{cr}\left[\left(1 - \frac{1}{k^2r^2}\right)\cos\psi - \frac{1}{kr}\sin\psi\right]. \qquad (3.11)$$

Here $\psi = kr - \omega t$. These expressions are valid at arbitrary distances from the current loop.

(3) For large distances ($kr \gg 1$), spherical Bessel functions can be changed by their asymptotic values

$$j_l(kr) \approx \frac{1}{kr}\cos\left(kr - \frac{l+1}{2}\pi\right)$$

$$n_l(kr) \approx \frac{1}{kr}\sin\left(kr - \frac{l+1}{2}\pi\right).$$

Then,

$$E_\phi = -H_\theta = \frac{\pi I_0 dk}{c}\frac{\cos\psi}{r} \times \sum_{n=0}^{\infty}(-1)^n \frac{4n+3}{(n+1)(2n+1)} P_{2n+1}^1 j_{2n+1}(kd) P_{2n+1}^1(0)$$

$$H_r = -\frac{2\pi I_0 d}{cr^2}\sin\psi \times \sum(-1)^n(4n+3)P_{2n+1} j_{2n+1}(kd) P_{2n+1}^1(0). \qquad (3.12)$$

The energy flux through the sphere of the radius r is

$$S_r = \frac{c}{4\pi}\int \mathrm{d}\Omega\, E_\phi H_\theta = \frac{2}{c}(I_0 kd\cos\psi)^2 \times \sum \frac{4n+3}{(n+1)(2n+1)}[j_{2n+1}(kd)P_{2n+1}^1(0)]^2.$$

The average energy lost for the period is

$$S_r = \frac{1}{c}(I_0 kd)^2 \sum \frac{4n+3}{(n+1)(2n+1)}[j_{2n+1}(kd)P_{2n+1}^1(0)]^2.$$

These expressions are valid for arbitrary kd.

3.1.1. Interaction of the current loop with an external EMF.

The interaction of current (3.6) with an external EMF is given by

$$U = -\frac{1}{c}\int \vec{j}_L \vec{A}_{ext}\,\mathrm{d}V. \qquad (3.13)$$

Since div $\vec{j}_L = 0$, the current density can be represented as

$$\vec{J}_L = \mathrm{curl}\,\vec{M}_L \qquad \vec{M}_L = I_L \vec{n}_z \Theta(d-\rho)\delta(z). \qquad (3.14)$$

Substituting this into (3.13) and integrating by parts, one obtains

$$U = -\frac{1}{c}\int \vec{M}_L \vec{H}_{ext}\,\mathrm{d}V.$$

For large distances compared with the loop radius d we have

$$U = -\frac{1}{c}\vec{H}_{ext}\int \vec{M}_L\,\mathrm{d}V = -\vec{\mu}\vec{H}_{ext}$$

where

$$\vec{\mu} = \frac{1}{c}\int \vec{M}\,\mathrm{d}V = \frac{1}{2c}\int \vec{r}\times\vec{j}\,\mathrm{d}V = \frac{I_L\pi d^2}{c}\vec{n}_z$$

coincides with the usual magnetic moment. These equations illustrate Ampere's hypothesis according to which the current loop is equivalent to the magnetic moment normal to it. When the radius d of the loop tends to zero,

$$\vec{M}_L \to I_L\pi d^2\vec{n}\delta^3(\vec{r}) \qquad \vec{J}_L = \mathrm{curl}\,\vec{M}_L$$

$$\delta^3(\vec{r}) = \delta(\rho)\delta(z)/2\pi\rho. \qquad (3.15)$$

Let now the dependence of this current flowing in the loop be $f_L(t)$, i.e.

$$\vec{J}_L = f_L(t)\,\mathrm{curl}\,\vec{n}_L\delta^3(\vec{r}) \qquad (3.16)$$

(the factor $\pi I_L d_L^2$ is absorbed into $f_L(t)$). Then, the EMF potentials and field strengths are given by

$$\vec{A}_L = -\frac{1}{c^2r^2}D_L(\vec{r}\times\vec{n}_L) \qquad \vec{E}_L = \frac{1}{c^3r^2}\dot{D}_L(\vec{r}\times\vec{n}_L)$$

$$\vec{H}_L = \frac{1}{c^3r}\left[\frac{(\vec{r}\vec{n}_L)}{r^2}\vec{r}F_L - \vec{n}_L G_L\right] \qquad (3.17)$$

where we put

$$D_L = D(f_L) = \dot{f}_l + \frac{c}{r}f_L$$

$$F_L = F(f_L) = \ddot{f}_l + \frac{3c}{r}\dot{f}_L + \frac{3c^2}{r^2}f_L$$

$$G_L = G(f_L) = \ddot{f}_l + \frac{c}{r}\dot{f}_L + \frac{c^2}{r^2}f_L. \qquad (3.18)$$

The arguments of the f_L functions entering into D_L, F_L and G_L are $t_r = t - r/c$; the dots above the f_L, D_L, F_L and G_L functions denote time derivatives. When f_L does not depend on time, one obtains the field of the elementary magnetic dipole

$$\vec{H}_L = \frac{p}{r^3}\left[3\vec{r}\frac{(\vec{r}\vec{n}_L)}{r^2} - \vec{n}_L\right]$$

of the power $p = f_L/c$. Obviously, equations (3.15)–(3.18) generalize (3.11) to arbitrary time dependences and orientations.

3.2. Historical remarks on the current loop

Equations (3.11) describing the EMF of the current loop in the long-wave limit may be found in many textbooks (see, e.g., Jackson [17], Stratton [21], Panofsky and Phillips [22]). In all probability, equations (3.9), valid for arbitrary distances and frequencies, may be found in journal papers; however, the author is unaware of any. It should be mentioned that even nowadays the EMFs of current loops are being investigated both theoretically [23] and experimentally [24].

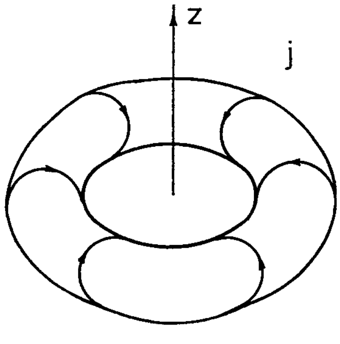

Figure 1. The poloidal current flowing on the surface of a torus.

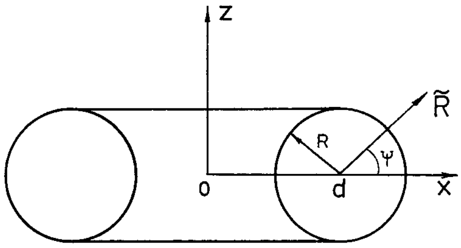

Figure 2. The coordinates $\tilde{R}$ and ψ parametrizing the torus.

3.3. Electromagnetic field of the toroidal solenoid

Consider the poloidal current flowing on the surface of a torus (figure 1)

$$\vec{j}_T = -\frac{gc}{4\pi}\vec{n}_\psi \frac{\delta(R-\tilde{R})}{d+\tilde{R}\cos\psi} \qquad \vec{n}_\psi = \vec{n}_z\cos\psi - \vec{n}_\rho \sin\psi. \tag{3.19}$$

The coordinates $\tilde{R}$, ψ and ϕ are related to the Cartesian coordinates as follows:

$$x = (d+\tilde{R}\cos\psi)\cos\phi \qquad y = (d+\tilde{R}\cos\psi)\sin\phi$$
$$z = \tilde{R}\sin\psi. \tag{3.20}$$

The condition $\tilde{R} = R$ defines the surface of a particular torus (figure 2). For $\tilde{R}$ fixed and ψ, ϕ varying, the points x, y, z given by (3.20) fill the surface of the torus $(\rho - d)^2 + z^2 = R^2$. The choice of $\vec{j}_0$ in the form of (3.19) is convenient, because in the static case a magnetic field H equals g/ρ inside the torus and vanishes outside it. In this case, g may also be expressed through either the magnetic flux Φ penetrating the torus or the total number N of turns in a toroidal winding and the current I in a particular turn:

$$g = \frac{\Phi}{2\pi(d-\sqrt{d^2-R^2})} = \frac{2NI}{c}.$$

Let the current in a TS winding periodically change with time: $\vec{j} = \vec{j}_0\exp(\mathrm{i}\omega t)$. Since

$$\vec{r}\times\vec{j}_T = \frac{gc}{4\pi}\delta(\tilde{R}-R)\vec{n}_\phi$$

and

$$\vec{r}\vec{j}_T = \frac{gcd\sin\psi}{4\pi}\frac{\delta(\tilde{R}-R)}{d+R\cos\psi}$$

one has

$$\mathrm{div}(\vec{r}\times\vec{j}) = 0 \qquad a_{lm}(M) = 0 \qquad a_{lm}(E) \neq 0.$$

Therefore,

$$\vec{A} = -\frac{4\pi\mathrm{i}k}{c}\sum \vec{A}_l(E)a_l(E)$$
$$\vec{H} = -\frac{4\pi k^2}{c}\sum \vec{A}_l(M)a_l(E)$$
$$\vec{E} = -\frac{4\pi k^2}{c}\sum \vec{A}_l(E)a_l(E) \tag{3.21}$$

($\vec{A}$ is the vector potential). From the facts that: (i) $\vec{r}\vec{H} = 0$ and (ii) $P\vec{A}_l(M) = (-1)^l\vec{A}_l(M)$, it follows [15, 17] that the radiation field of a TS is of electric type.

The electric form factor $a_l(E)$ for the radiating TS is equal to

$$a_l(E) = \frac{1}{4}gcdRk\sqrt{\frac{2l+1}{\pi l(l+1)}}I_l$$
$$I_l = \int_0^{2\pi} j_l(ky)P_l(x)\sin\psi\,\mathrm{d}\psi \tag{3.22}$$

where $y = [d^2 + R^2 + 2dR\cos\psi]^{1/2}$ and $x = R\sin\psi/y$. It easy to check that $a_l(E) = 0$ for l even. Let the current time dependence be $\cos\omega t$. Then the EMF is given by the real parts of $\vec{A}$, $\vec{E}$ and $\vec{H}$:

$$A_\theta = \frac{gdRk^2}{2}\sum\frac{1}{l(l+1)}I_lP_l^1\{[(l+1)j_{l-1} - lj_{l+1}]$$
$$\times\sin\omega t - [(l+1)n_{l-1} - ln_{l+1}]\cos\omega t\}$$
$$A_r = \frac{gdRk}{2r}\sum(2l+1)I_lP_l(j_l\sin\omega t - n_l\cos\omega t)$$
$$H_\phi = \frac{gdRk^3}{2}\sum\frac{2l+1}{l(l+1)}I_lP_l^1(n_l\cos\omega t - j_l\sin\omega t)$$
$$E_\theta = -\frac{gdRk^3}{2}\sum\frac{1}{l(l+1)}I_lP_l^1\{[(l+1)j_{l-1} - lj_{l+1}]$$
$$\times\cos\omega t + [(l+1)n_{l-1} - ln_{l+1}]\sin\omega t\}$$
$$E_r = -\frac{gdRk^2}{2r}\sum(2l+1)I_lP_l(j_l\cos\omega t + n_l\sin\omega t). \tag{3.23}$$

The number of $a_l(E)$ contributing to the sums in (3.23) is the same as for current loop: it should be slightly greater than $0.7kd$.

Consider the following particular cases.

(1) In the static limit ($k \to 0$) one obtains

$$I_l \to \frac{k^l}{(2l+1)!!}C_l \qquad a_l(E) \to \frac{gcdRk^{l+1}}{4(2l+1)!!}\sqrt{\frac{2l+1}{\pi l(l+1)}}C_l$$

$$T_{lm} = \delta_{m0}T_l \qquad T_l = \frac{gdR\sqrt{l}}{2(l+1)}C_l$$

$$C_l = \int_0^{2\pi} y^lP_l\left(\frac{R\sin\psi}{y}\right)\sin\psi\,\mathrm{d}\psi \tag{3.24}$$

where T_{lm} is the same as in (3.5). This integral can be taken in a closed form. We give its value only for $l = 1$

$$C_1 = \pi R \qquad a_1(E) = \frac{\pi gdR^2k^2c}{4\sqrt{6\pi}}.$$

The EMF strengths of the TS decrease as k^2

$$H_\phi \sim -gdRk^2\sum\frac{1}{l(l+1)}\frac{1}{r^{l+1}}C_lP_l^1$$

$$E_\theta \sim -\frac{gdRk^2}{2}ct\sum\frac{C_l}{l+1}\frac{1}{r^{l+2}}P_l^1$$

$$E_r \sim \frac{gdRk^2}{2}ct\sum C_lP_l\frac{1}{r^{l+2}}. \tag{3.25}$$

On the other hand, the vector potential of a TS does not vanish in the static limit

$$A_\theta \to -\frac{gdR}{2}\sum\frac{C_l}{l+1}P_l^1\frac{1}{r^{l+2}}$$

$$A_r \to \frac{gdR}{2}\sum\frac{1}{r^{l+2}}C_lP_l. \tag{3.26}$$

The linear time dependence in $\vec{E}$ (for $\omega t \ll 1$) arises when one differentiates the $\cos\omega t$ term in $\vec{A}$ and then let ω go to zero. For the infinitely thin TS ($R \ll d$), C_l is reduced to

$$C_{2n+1} = \pi Rd^{2n}(-1)^n\frac{(2n+1)!!}{2^n n!}.$$

(2) Infinitely small toroidal solenoid ($kd \ll 1$). Obviously, only the $l = 1$ term contributes to sums in (3.23)

$$I_1 = \frac{\pi kR}{3} \qquad a_1(E) = \frac{\pi gdR^2k^2c}{4\sqrt{6\pi}}$$

$$E_r = \frac{\pi gdR^2k^2}{2r^2}\cos\theta\left[\cos\psi - \frac{1}{kr}\sin\psi\right]$$

$$E_\theta = \frac{\pi gdR^2k^3}{4r}\sin\theta\left[\sin\psi\left(1-\frac{1}{k^2r^2}\right)+\frac{1}{kr}\cos\psi\right]$$

$$H_\phi = \frac{\pi gdR^2k^3}{4r}\sin\theta\left[\sin\psi + \frac{1}{kr}\cos\psi\right]. \tag{3.27}$$

For estimations, let the major radius d of a TS be 10 cm. We rewrite the condition $kd \ll 1$ in the wavelength language

$$\frac{2\pi d}{\lambda} \approx \frac{60}{\lambda} \ll 1.$$

This means that equations (3.27) will work for $\lambda \geqslant 5$m.

(3) Infinitely thin toroidal solenoid ($R \ll d$). Taking into account the fact that

$$P_{2n+1}(x) \to -P_{2n+1}^1(0)x \qquad \text{for} \quad x \to 0$$

one obtains

$$I_{2n+1} = -\frac{R}{d}P_{2n+1}^1(0)D_{2n+1}$$

$$D_{2n+1} = \int_0^{2\pi} j_{2n+1}(ky)\sin^2\psi\,\mathrm{d}\psi$$

$$a_{2n+1}(E) = -\frac{1}{4}gcR^2k\sqrt{\frac{4n+3}{\pi(2n+1)2(n+1)}}P_{2n+1}^1(0)D_{2n+1}. \tag{3.28}$$

For $R \ll d$ (but for arbitrary kd and kR) D_{2n+1} can be taken in a closed form (see the appendix):

$$D_{2n+1} = \pi\{J_0(kR)j_{2n+1}(kd) - \tfrac{1}{2}J_2(kR)[j_{2n+3}(kd)+j_{2n-1}(kd)]\}. \tag{3.29}$$

If, in addition, $kR \ll 1$, then

$$D_{2n+1} = \pi j_{2n+1}(kd)$$

and

$$a_{2n+1}(E) = -\frac{\pi}{4}gcR^2k\sqrt{\frac{4n+3}{\pi(2n+1)2(n+1)}}P_{2n+1}^1(0)j_{2n+1}(kd). \tag{3.30}$$

On the other hand, if $kR \gg 1$, then

$$D_{2n+1} = \frac{2}{kd}\sqrt{\frac{2\pi}{kR}}\cos\left(kR-\frac{\pi}{4}\right)[(n+1)j_{2n+2}(kd)+nj_{2n}(kd)] \tag{3.31}$$

(we cannot substitute instead of $J_{2n}(kd)$ and $J_{2n+2}(kd)$ their asymptotic values, since the presence of $J_{2n}(kd)$ and $J_{2n+2}(kd)$ guarantees the convergence of electromagnetic strengths, (3.23)).

For $kd \gg 1$, equations (3.27) are not applicable. For example, for $d = 10$ cm and $\lambda = 1$cm, $kd \approx 60$. The possible outcome is to take the minor radius of a TS as small as possible. Equations (3.23) with $a_l(E)$ given by (3.28) and (3.29) are valid for arbitrary frequencies if $R \leqslant 2$ cm (for $d = 10$ cm). The advantage of electric form factors (3.28) and (3.29) is that they do not involve integration, which is very cumbersome for high frequencies.

(4) Large distances ($kr \gg 1$). Then,

$$E_\theta = H_\phi = -\frac{gdRk^2}{4r}\sin\psi \times\sum(-1)^n\frac{4n+3}{(2n+1)(n+1)}I_{2n+1}P_{2n+1}^1 \tag{3.32}$$

$$E_r = \frac{gdRk}{2r^2}\cos\psi\sum(4n+3)(-1)^nI_{2n+1}P_{2n+1}.$$

The energy flux through the sphere of the radius r is

$$S_r = \frac{c}{4\pi}r^2\int\mathrm{d}\Omega\,E_\theta H_\phi = c\left(\frac{gdRk^2\sin\psi}{2}\right)^2\sum\frac{4n+3}{2(n+1)(2n+1)}I_{2n+1}^2. \tag{3.33}$$

Correspondingly, the average energy lost for the period is

$$S_r = \frac{c}{2}\left(\frac{gdRk^2}{2}\right)^2\sum\frac{4n+3}{2(n+1)(2n+1)}I_{2n+1}^2.$$

3.3.1. *Interaction of a TS with an external EMF.*

The interaction of a TS with external EMF is given by

$$U = -\frac{1}{c}\int\vec{j}_T\vec{A}_{ext}\,\mathrm{d}V. \tag{3.34}$$

Since $\operatorname{div}\vec{j}_T = 0$, the poloidal current (3.19) flowing on the surface of the torus can be represented in the form [25]

$$\vec{j}_T = \operatorname{curl}\vec{M} \qquad \operatorname{div}\vec{M} = 0$$

$$\vec{M} = \vec{n}_\phi\frac{gc}{4\pi\rho}\Theta\left(R-\sqrt{(\rho-d)^2+z^2}\right) \qquad \operatorname{div}\vec{M} = 0. \tag{3.35}$$

That is, the magnetization $\vec{M}$ has only the azimuthal component and differs from zero only inside the torus (middle

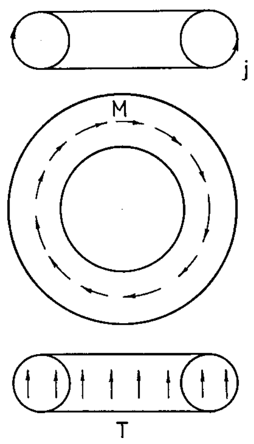

Figure 3. The poloidal current $\vec{j}$ flowing on the surface of a torus is equivalent to the magnetization $\vec{M}$ confined to the interior of the torus and to the toroidization $\vec{T}$ directed along the axis of the torus.

part of figure 3). Since $\operatorname{div}\vec{M} = 0$, the magnetization $\vec{M}$, in its turn, can be written as

$$\vec{M} = \operatorname{curl}\vec{T} \qquad \operatorname{div}\vec{T} \neq 0 \tag{3.36}$$

where

$$\vec{T} = \vec{n}_z T$$
$$T = \frac{gc}{4\pi}\left[\Theta\left(d - \sqrt{R^2 - z^2} - \rho\right)\ln\frac{d + \sqrt{R^2 - z^2}}{d - \sqrt{R^2 - z^2}} + \Theta\left(d + \sqrt{R^2 - z^2} - \rho\right)\Theta\left(\rho - d + \sqrt{R^2 - z^2}\right) \times \ln\frac{d + \sqrt{R^2 - z^2}}{\rho}\right]. \tag{3.37}$$

Thus, T differs from zero in two space regions (see the lower part of figure 3) as follows.

(a) Inside the torus hole defined as $0 \leqslant \rho \leqslant d - \sqrt{R^2 - z^2}$, where T does not depend on ρ

$$T_z = \frac{gc}{4\pi}\ln\frac{d + \sqrt{R^2 - z^2}}{d - \sqrt{R^2 - z^2}}. \tag{3.38}$$

(b) Inside the torus itself ($d - \sqrt{R^2 - z^2} \leqslant \rho \leqslant d + \sqrt{R^2 - z^2}$) where

$$T = \frac{gc}{4\pi}\ln\frac{d + \sqrt{R^2 - z^2}}{\rho}. \tag{3.39}$$

In other space regions, $T = 0$. Therefore,

$$\vec{j}_T = \operatorname{curl}\operatorname{curl}\vec{T} \qquad \operatorname{div}\vec{T} \neq 0. \tag{3.40}$$

Substituting (3.40) into (3.34), one obtains

$$U = -\frac{1}{c^2}\int \dot{\vec{E}}\vec{T}\,\mathrm{d}V$$

(the dot above $\vec{E}$ denotes time derivative). For distances large compared with the large radius of a TS

$$U = -\frac{1}{c^2}\dot{\vec{E}}\int \vec{T}\,\mathrm{d}V. \tag{3.41}$$

Despite the fact that T is rather complicated, the volume integral looks very simple

$$\int \vec{T}\,\mathrm{d}V = \vec{n}_z\frac{\pi cgdR^2}{4}. \tag{3.42}$$

Physically, equations (3.35), (3.36) and (3.40) mean that the poloidal current $\vec{j}$ given by (3.35) is equivalent (i.e. produces the same magnetic field) to the toroidal tube with magnetization $\vec{M}$ defined by (3.36) and to toroidization $\vec{T}$ given by (3.37). This is illustrated in figure 3. Obviously, these equations generalize Ampere's hypothesis.

Now let the minor radius R of a torus tend to zero (this corresponds to an infinitely thin torus). Then, the second term in (3.37) drops out, while the first one reduces to

$$T \to \frac{gc}{2\pi d}\Theta(d - \rho)\sqrt{R^2 - z^2}. \tag{3.43}$$

For infinitesimal R

$$\sqrt{R^2 - z^2} \to \tfrac{1}{2}\pi R^2\delta(z).$$

Therefore, in this limit,

$$\vec{j} = \operatorname{curl}\operatorname{curl}\vec{T} \qquad \vec{T} = \vec{n}_z\frac{gcR^2}{4d}\delta(z)\Theta(d - \rho). \tag{3.44}$$

That is, the vector $\vec{T}$ is confined to the equatorial plane of a torus and is perpendicular to it. Let now $d \to 0$ (in addition to $R \to 0$). Then,

$$\frac{1}{d}\Theta(d - \rho) \to \frac{d}{2\rho}\delta(\rho)$$

and the current of an elementary (i.e. infinitely small) TS is

$$\vec{j} = \operatorname{curl}\operatorname{curl}\vec{T} \qquad \vec{T} = \tfrac{1}{4}\pi cgdR^2\delta^3(\vec{r})\vec{n}_z. \tag{3.45}$$

Let now the dependence of the current flowing in the toroidal solenoid be $f_T(t)$, i.e.

$$\vec{J}_T = f_T(t)\operatorname{curl}\vec{n}_T\delta^3(\vec{r}). \tag{3.46}$$

(the factor $\frac{1}{4}\pi cgd_T R^2$ is included in $f_T(t)$). Then, the EMF potentials and field strengths are given by

$$\vec{A}_T = \frac{1}{c^3 r}\left[-\vec{n}_T G_T + \frac{1}{r^2}\vec{r}(\vec{r}\vec{n}_T)F_T\right]$$
$$\vec{E}_T = \frac{1}{c^4}r\left[\vec{n}_T\dot{G}_T - \frac{1}{r^2}\vec{r}(\vec{r}\vec{n}_T)\dot{F}_T\right]$$
$$\vec{H}_T = \frac{1}{4c^3 r}(\vec{r}\times\vec{n}_T)\ddot{D}_T \tag{3.47}$$

where the functions $D_T = D(f_T)$, $F_T = F(f_T)$ and $G_T = G(f_T)$ are defined by (3.18). When f_T is independent of t, the EMF strengths are zero, only the vector potential survives

$$\vec{A}_T = -\frac{1}{4cr^3}f_T\left[\vec{n}_T - \frac{3}{r^2}\vec{r}(\vec{r}\vec{n}_T)\right].$$

Clearly, (3.47) generalizes (3.27) for arbitrary time dependences and orientations.

3.3.2. Toroidal solenoids with a more realistic winding. Usually, the toroidal coil twists along the torus surface having not only a $\vec{n}_\psi$ component, but also a $\vec{n}_\phi$ component parallel to the torus equatorial line. Then, the total density is given by

$$\vec{j} = \cos\alpha\, \vec{j}_T + \sin\alpha\, \vec{j}_L \tag{3.48}$$

where $\vec{j}_L$ and $\vec{j}_T$ are given by (3.6) and (3.19), respectively, and α is the inclination angle of the current $\vec{j}$ towards the vector $\vec{n}_\psi$.

Since the EMF is a linear function of the current density, it is given by

$$\vec{E} = \cos\alpha\, \vec{E}_T + \sin\alpha\, \vec{E}_L \qquad \vec{H} = \cos\alpha\, \vec{E}_H + \sin\alpha\, \vec{H}_L \tag{3.49}$$

where $\vec{E}_L$, $\vec{H}_L$ and $\vec{E}_T$, $\vec{H}_T$ are the EMFs of the current loop and TS given by (3.9) and (3.23), respectively. For $\alpha = 0$ and $\alpha = \pi/2$ the EMF (3.49) is transformed either into an EMF (3.23) of a TS or an EMF (3.9) of a current loop.

However, if there is a need to create pure toroidal EMF (3.23), then one should somehow compensate the $\vec{n}_\phi$ component of $\vec{j}$. For this, after finishing the toroidal winding (3.48) (i.e. when the last turn of the toroidal winding meets the first one), one closed turn lying in the equatorial torus plane and having the direction opposite to $\vec{n}_\phi$ should be added. Another possibility is to use a winding consisting of an even number of layers. If the directions of coils in the even and odd layers differ by the sign of α, then $\vec{j}_\phi$ current components of even and layers compensate each other and only the $\vec{j}_\psi$ component survives.

3.4. Historical remarks on TSs

TSs play an important role in physics and technology. As the simplest three-dimensional topologically non-trivial objects, they have been used for the experimental verification of the Aharonov–Bohm effect [26]. The corresponding calculations were performed in [27]. They possess a number of non-trivial characteristics such as toroidal [18, 28] and 'hidden' [29] moments. Exact vector potentials of finite static TSs were evaluated by Luboshitz and Smorodinsky [30], in a non-standard gauge, and in [31], in a Coulomb gauge. Similarly to the static magnetic TSs outside which the EMF strengths disappear, but the magnetic vector potential differs from zero, there are electric TSs outside which the EMF strengths are zero but non-trivial *electric* vector potentials differ from zero [25, 32]. Furthermore, there exists the toroidal Aharonov–Casher effect which describes quantum (not classical) scattering of toroidal dipoles by the electric charge [33].

Turning to TSs with time-dependent currents, one should mention two papers by Page [34]. However, his EMF strengths were presented in the integral form, unsuitable for practical applications. The EMF of TSs for a number of time dependences were studied in [35]. Unfortunately, the most interesting case of a periodical current was considered for a very special case of an infinitely small TS. The multipole expansion of the EMF for a TS with periodical current was given in [19, 36]. However, these presentations were too schematic, without practical applications. Equations (3.47) for the EMF of an infinitely small TS was earlier been obtained by Nevessky [37] and, also, in [38]. Their generalization for more complicated toroidal configurations is given in [39]. In the same reference, as well as in [25], the charge–current toroidal configurations were found outside which non-trivial (that is unremovable by a gauge transformation) time-dependent electromagnetic potentials were different from zero despite the vanishing EMF strengths. This makes possible the performance of experiments investigating the time-dependent Aharonov–Bohm effect. All these studies are summarized in [40].

What is new in this section? It seems that general equations (3.23) defining EMFs of TS and corresponding particular cases (3.24)–(3.33) were not considered previously.

We briefly enumerate the applications of TSs as follows.

(a) Toroidal transformers are very effective since the leakages of the EMF into the surrounding space are very small.
(b) TSs are widely used in modern accelerators. Being placed along the circumference, they generate electromagnetic field concentrated inside the torus holes (see e.g. [25], where the exactly soluble configuration of a TS producing a time-dependent EMF confined to the interior of a circular tube was considered).
(c) According to [41] 'Air-cored toroidal inductors are used in power electronic circuits because they are relatively easy to make, they do not saturate and they do not produce troublesome external magnetic fields.'
(d) Finally, one should mention Birkeland's electromagnetic gun (see, e.g., [42]) in which the set of toroidal solenoids are used for the acceleration of an iron bullet. The modern version of Birkeland's gun is realized in US *Star Wars* programme, officially known as the Strategic Defence Initiative.

4. EMF of an electric dipole

Consider two point charges at the points $\pm a_d\vec{n}$. Their charge density is given by

$$\rho_d = e[\delta^3(\vec{r} - a_d\vec{n}) - \delta^3(\vec{r} + a_d\vec{n})].$$

For an infinitely small dipole, this takes the form

$$\rho_d = -2ea(\vec{n}\vec{\nabla})\delta^3(\vec{r}) \qquad \vec{\nabla}_i = \frac{\partial}{\partial x_i}.$$

Now let the charge density depend on time

$$\rho_d = f(t)(\vec{n}\vec{\nabla})\delta^3(\vec{r})$$

(factor $-2ea$ is included in $f(t)$). The corresponding current density is given by

$$\vec{j}_d = -\dot{f}(t)\vec{n}\delta^3(\vec{r}). \tag{4.1}$$

The following EMF strengths correspond to these densities:

$$\vec{H}_d = \frac{1}{c^2r^2}(\vec{r}\times\vec{n})\dot{D}_d$$

$$\vec{E}_d = \frac{1}{c^2r}\left[\vec{n}G_d - \frac{1}{r^2}(\vec{n}\vec{r})\vec{r}F_d\right]. \tag{4.2}$$

Now let the time dependence of the charge density be $\cos\omega t$:

$$\rho_d = -2ea_d\cos\omega t(\vec{n}\vec{\nabla})\delta^3(\vec{r})$$
$$\vec{j}_d = -2ea_d\omega\sin\omega t\vec{n}\delta^3(\vec{r}). \tag{4.3}$$

For the unit vector $\vec{n}$ along the z axis, one obtains

$$H_{d\phi} = -\frac{2ea_dk^2}{r}\sin\theta\left(\cos\psi - \frac{\sin\psi}{kr}\right)$$
$$E_{d\theta} = -\frac{2ea_dk^2}{r}\sin\theta\left[\cos\psi\left(1-\frac{1}{k^2r^2}\right) - \frac{\sin\psi}{kr}\right]$$
$$E_d^r = \frac{4ea_dk}{r^2}\cos\theta\left(\sin\psi + \frac{1}{kr}\cos\psi\right) \qquad \psi = kr-\omega t. \tag{4.4}$$

In the static limit ($k\to 0$) one obtains the field of an electric dipole

$$E_{d\theta}\to\frac{2a_de}{r^3}\sin\theta \quad E_{dr}\to\frac{4ea_d}{r^3}\cos\theta \quad H_{d\phi}\to 0.$$

For the oscillating electric dipole with a finite a_d, oriented along the z axis

$$\rho_d = \mathrm{i}\exp(\mathrm{i}\omega t)\rho_{d0}$$
$$\rho_{d0} = \frac{e}{2\pi a_d^2\sin\theta}\delta(r-a_d)[\delta(\theta)-\delta(\pi-\theta)]$$
$$\vec{j}_d = \vec{n}_r j_d \qquad j_d = -\omega\exp(\mathrm{i}\omega t)j_{d0}$$
$$j_{d0} = \frac{e}{2\pi r^2\sin\theta}\Theta(a_d-r)[\delta(\theta)-\delta(\pi-\theta)] \tag{4.5}$$

If we desire to obtain, in the static limit, the static electric density ρ_{d0}, we should take, at the end of all calculations, the imaginary parts of the EMF strengths (since ρ_d in (4.5) contains the imaginary unit factor i). It turns out that $a_l^m(M)=0$, i.e. only the electric form factors with l odd contribute to the EMF strengths,

$$a_l^m(E) = \delta_{m0}a_l(E) \qquad a_l(E) = -2ec\sqrt{\frac{1}{l(l+1)}}F_l(ka_d) \tag{4.6}$$
$$F_l(ka_d) = \int_0^{ka_d} j_l(x)x\,\mathrm{d}x + ka_d\frac{(l+1)j_{l-1}(ka_d)-lj_{l+1}(ka_d)}{2l+1}.$$

For $ka_d\to 0$ this reduces to

$$F_l\to\frac{l+1}{(2l+1)!!}(ka_d)^l.$$

Taking the imaginary parts of the EMF strengths (3.18) with $a_l(E)$ given by (4.6), one obtains

$$H_\phi = -2ek^2\sum\frac{2l+1}{l(l+1)}(\cos\omega t j_l + \sin\omega t n_l)P_l^1F_l(ka_d)$$
$$E_\theta = -2ek^2\sum\frac{1}{l(l+1)}\{\cos\omega t[(l+1)n_{l-1}-ln_{l+1}]$$
$$-\sin\omega t[(l+1)j_{l-1}-lj_{l+1}]\}P_l^1F_l(ka_d)$$
$$E_r = -\frac{2ek}{r}\sum(2l+1)(\cos\omega t n_l - \sin\omega t j_l)P_lF_l(ka_d). \tag{4.7}$$

We evaluate the square bracket entering into the definition of the toroidal moment (see the last line in (3.4)) for the electric dipole charge–current density given by (4.5):

$$(l+3)\int r^{l+2}Y_{lm}^*\,\mathrm{div}\,\vec{j}_d\,\mathrm{d}V + 2(2l+3)\int r^lY_{lm}^*(\vec{r}\vec{j}_d)\,\mathrm{d}V$$
$$= \delta_{m0}\frac{2e\omega l(l+1)}{(l+2)}a_d^{l+2} \tag{4.8}$$

(factor $\exp(\mathrm{i}\omega t)$ is omitted).

4.1. Interaction of an electric dipole with an external EMF

Substituting the charge–current densities of the elementary electric dipole

$$\rho_d = f(t)(\vec{n}\vec{\nabla})\delta^3(\vec{r}-\vec{r}_d) \qquad \vec{j}_d = -\dot{f}(t)\vec{n}\delta^3(\vec{r}-\vec{r}_d)$$

into the expression for the interaction energy

$$U = \int\left[\rho_d(\vec{r})\Phi_{ext}(\vec{r}) - \frac{1}{c}\vec{j}_d(\vec{r})\vec{A}_{ext}(\vec{r})\right]\mathrm{d}V$$

one obtains

$$U = -f_d(t)(\vec{n}\vec{\nabla})\Phi_{ext}(\vec{r}_d) + \frac{1}{c}\dot{f}_d(t)\vec{n}\vec{A}_{ext}(\vec{r}_d). \tag{4.9}$$

Let the external EMF be the field of a TS with a constant current in its winding. Then, outside the TS, $\Phi_{ext}=0$, $\vec{E}_{ext}=0$, $\vec{H}_{ext}=0$, $\vec{A}_{ext}\neq 0$ and

$$U = -\frac{1}{c}\dot{f}_d(t)\vec{n}\vec{A}_{ext}(\vec{r}_d). \tag{4.10}$$

It is surprising enough that the interaction energy differs from zero in the space region where $\vec{E}_{ext}=\vec{H}_{ext}=0$. Despite the fact that EMF strengths vanish outside the static TS, the vector potential $\vec{A}$ cannot be eliminated by a gauge transformation everywhere in this region. This is due to the fact that $\int\vec{A}\,\mathrm{d}\vec{s}$ along any closed path passing through the TS hole, is equal to the magnetic flux inside the TS. However, the space region where $\vec{A}$ differs from zero depends on the gauge choice (see, e.g., [40]). On the other hand, the interaction energy (4.10) should not depend on the gauge choice. The origin of this inconsistency is unclear for us.

4.2. Historical remarks on electric dipoles

The EMF of an electric dipole is analysed almost in any textbook on classical electrodynamics. However, all of them are limited to the long-wave limit, expressions (4.2) and (4.4). We did not see the general equations (4.7). This is due to the fact that for the typical wavelength of the short-wave range ($\lambda = 25$ m), $ka_d \ll 1$ for $a_d \sim 10$ cm. In this case equations (4.7) are reduced to (4.2) and (4.4). However, in the microwave region equations (4.7) should be used.

5. More complicated elementary toroidal sources

In this section we give, without derivation, the EMFs of more complicated toroidal sources obtained earlier in [39]. They are needed for the evaluation of integrals entering in the Lorentz and Feld–Tai theorems. Unfortunately, their omission makes the text unreadable. Consider the hierarchy of a TS each turn of which is again a TS. The simplest of them is the usual TS obtained by the replacement of a single turn, representing the current loop, by the infinitely thin TS. We denote this TS by TS_1 (the initial current loop will be denoted by TS_0). The next-in-complexity case is obtained when each turn of TS_1 is replaced by an infinitely thin toroidal solenoid ts_1 with the time-dependent current in its winding. The thus obtained current configuration denoted by TS_2 is shown in figure 4. We see on it the poloidal current $\vec{j}$ flowing on the surface

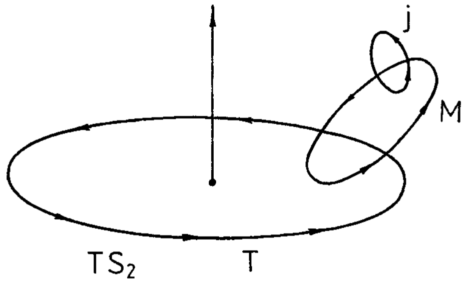

Figure 4. A toroidal source of the second order is obtained if instead of each particular turn of a usual TS, a new infinitely thin TS ts is substituted with the current $\vec{j}$ in its winding; it generates the magnetization $\vec{M}$ covering the surface of the original TS and directed along its meridians. The complete magnetization from all ts generates the closed tube of toroidal moments T filling the interior of the original TS and generating in turn the second-order toroidal moment shown by the vertical arrow.

of a particular torus ts_1. Only one particular turn with the current $\vec{j}$ and only the central line of ts_1 are shown (for the torus $(\rho - d)^2 + z^2 = R$, the central line is defined as $\rho = d$, $z = 0$). The arising time-dependent magnetization (due to the current $\vec{j}$ flowing in ts_1) coincides with the central line of ts_1 and lies on the surface of TS_1, in its meridional plane. Since there are many turns in TS_1, (each of them is the same as ts_1), the superposition of their magnetizations gives the overall magnetization $\vec{M}$, filling the surface of TS_1 (see figure 1 or the upper part of figure 3, where $\vec{j}$ now means $\vec{M}$). This distribution of magnetization is equivalent to the closed chain of toroidal moments $\vec{T}$ aligned along the central line of TS_1 (see the middle part of figure 3, where $\vec{M}$ now means $\vec{T}$). The closed chain of toroidal moments leads to the appearance of a higher-order toroidal moment shown in figure 4 by the vertical arrow. When the dimensions of this, just obtained, configuration TS_2 tend to zero, we obtain (see [25, 39])

$$\vec{j}_2 = f_2(t)\,\mathrm{curl}^{(3)}(\vec{n}\delta^3(\vec{r})) \qquad \mathrm{curl}^{(3)} = \mathrm{curl}\cdot\mathrm{curl}\cdot\mathrm{curl}. \tag{5.1}$$

The corresponding vector potential and field strengths are given by

$$\vec{A}_2 = \frac{1}{c^4 r^2} D_2^{(2)}(\vec{r}\times n) \qquad \vec{E}_2 = -\frac{1}{c^5 r^2}\, D_2^{(3)}(\vec{r}\times n)$$

$$\vec{H}_2 = \vec{n}\frac{1}{c^5 r} G_2^{(2)} - \frac{1}{c^5 r^3}\vec{r}(\vec{r}\vec{n})F_2^{(2)}. \tag{5.2}$$

Here the subscripts on the D, F and G functions mean that they depend on the f function with the given index, while the superscript denotes the time derivative of the order equal to the superscript. For example,

$$D_m^{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}t^n} D(f_m).$$

The argument of f functions is $t - r/c$. By comparing (5.1) and (5.2) with (3.16) and (3.17) we conclude that for the current configurations TS_0 and TS_2 the electromagnetic fields coincide everywhere except for the origin if the following relation between the time-dependent intensities is fulfilled: $f_2^{(2)} = -f_0/c^2$. This means, in particular, that the EMF of the static magnetic dipole ($f_0 =$ constant) coincides with that of the current configuration TS_2 if the current in it quadratically varies with time ($f_2 = -f_0 c^2 t^2/2$). It follows from this that the magnetic field of the usual magnetic dipole can be compensated everywhere (except for the origin) by the time-dependent current flowing in TS_2.

Now we are able to write out the EMF for the point-like toroidal configuration of arbitrary order. Let

$$\vec{j}_m = f_m(t)\,\mathrm{curl}^{(m+1)}(\vec{n}\delta^3(\vec{r})). \tag{5.3}$$

We consider even and odd m separately.

5.1. *Toroidal configurations of even order*

Let m be even ($m = 2k$, $k \geqslant 0$). Then

$$\vec{A}_{2k} = (-1)^{k+1}\frac{1}{c^{2k+2}r^2}\, D_{2k}^{(2k)}(\vec{r}\times n)$$

$$\vec{E}_{2k} = (-)^k \frac{1}{c^{2k+3}r^2} D_{2k}^{(2k+1)}(\vec{r}\times n)$$

$$\vec{H}_{2k} = (-1)^k \frac{1}{c^{2k+3}}\left[\frac{1}{r^3}\vec{r}(\vec{r}\vec{n})F_{2k}^{(2k)} - \vec{n}\frac{1}{r}G_{2k}^{(2k)}\right]. \tag{5.4}$$

The distribution of the radial energy flux on the sphere of radius r is given by

$$S_r = \frac{c}{4\pi}(\vec{E}\times\vec{H})_r = \frac{\sin^2\theta}{4\pi c^{4k+5} r^2} D_{2k}^{2k+1} G_{2k}^{2k}.$$

Here θ is the angle between the symmetry axis $\vec{n}$ and a particular point on the sphere. The total energy flux through this sphere is

$$r^2 \int S_r\, \mathrm{d}\Omega = \frac{2}{3c^{4k+5}} D_{2k}^{2k+1} G_{2k}^{2k}.$$

The interaction of the even toroidal source with the external EMF is given by

$$U = -\frac{f_{2k}}{c}\int \mathrm{d}V\, \vec{A}_{ext}\,\mathrm{curl}^{2k+1}(\vec{n}\delta^3(\vec{r}-\vec{r}_s))$$
$$= (-1)^{k+1}\frac{f_{2k}}{c^{2k+1}}(\vec{n}\vec{H}_{ext}^{(2k)}) \tag{5.5}$$

where the external magnetic field is taken at the position of a point-like toroidal source.

5.2. *Toroidal configurations of odd order*

On the other hand, for m odd ($m = 2k+1$, $k \geqslant 0$)

$$\vec{A}_{2k+1} = (-1)^k \frac{1}{c^{2k+3}}\left[\frac{1}{r^3}\vec{r}(\vec{r}\vec{n})F_{2k+1}^{(2k)} - \vec{n}\frac{1}{r}G_{2k+1}^{(2k)}\right]$$

$$\vec{E}_{2k+1} = (-1)^{k+1}\frac{1}{c^{2k+4}}\left[\frac{1}{r^3}\vec{r}(\vec{r}\vec{n})F_{2k+1}^{(2k+1)} - \vec{n}\frac{1}{r}G_{2k+1}^{(2k+1)}\right]$$

$$\vec{H}_{2k+1} = (-1)^k \frac{1}{c^{2k+4}r^2} D_{2k+1}^{(2k+2)}(\vec{r}\times n)$$

$$S = \frac{2}{3c^{4k+7}} G_{2k+1}^{(2k+1)} D_{2k+1}^{(2k+2)}. \tag{5.6}$$

The distribution of the radial energy flux on the sphere of radius r is given by

$$S_r = \frac{c}{4\pi}(\vec{E}\times\vec{H})_r = \frac{\sin^2\theta}{4\pi c^{4k+7} r^2} D_{2k+1}^{2k+2} G_{2k+1}^{2k+1}.$$

The total energy flux through this sphere is

$$r^2\int S_r\,\mathrm{d}\Omega = \frac{2}{3c^{4k+7}}D_{2k+1}^{2k+2}G_{2k+1}^{2k+1}.$$

The interaction of the odd toroidal source with the external EMF is given by

$$U = -\frac{f_{2k+1}}{c}\int \mathrm{d}V\,\vec{A}_{ext}\,\mathrm{curl}^{2k+2}(\vec{n}\delta^3(\vec{r}-\vec{r}_s))$$
$$= (-1)^{k+1}\frac{f_{2k+1}}{c^{2k+2}}(\vec{n}\vec{E}_{ext}^{(2k+1)}). \qquad (5.7)$$

Again, the external electric field is taken at the position of a point-like toroidal source.

5.3. Short resumé of this section

We see that there are two branches of toroidal point-like currents generating essentially different electromagnetic fields. A representative of the first branch is the usual magnetic dipole. The electromagnetic field of the kth member of this family reduces to that of the circular current if the time dependences of these currents are properly adjusted

$$f_{2k}^{(2k)} = (-1)^k f_0(t)/c^{2k} \qquad (k \geqslant 0). \qquad (5.8)$$

We remember that the lower index of the f functions selects a particular member of the first branch, while the upper one means the time derivative.

The representative of the second branch is the elementary TS. Again, the electromagnetic fields of this family are the same if the time dependences of currents are properly adjusted

$$f_{2k+1}^{(2k)} = (-1)^k f_1(t)/c^{2k} \qquad (k \geqslant 0). \qquad (5.9)$$

From the equations defining the energy flux it follows that for high frequencies, the toroidal emitters of the higher order are more effective (as the time derivatives of the higher orders contribute to the energy flux). They may be used in the same way as usual frequency modulation transmitters. Namely, the EMF of high frequency carries the energy. It is modulated by the low-frequency EMF carrying the information. The resulting signal is decoded in the receiver, its high frequency is removed, while its low-frequency part comes to our ears.

From the classical electrodynamics it is known [15, 17] that there are two types of radiation. For the multipole radiation of magnetic type $\vec{r}\vec{E} = 0$ and $\vec{r}\vec{H} \neq 0$, while for radiation of the electric type should be $\vec{r}\vec{H} = 0$ and $\vec{r}\vec{E} \neq 0$. It follows from (5.4) that $\vec{r}\vec{E}_{2k} = 0$ and $\vec{r}\vec{H}_{2k} \neq 0$. Thus, radiation fields of the time-dependent currents flowing in a circular turn and in toroidal emitters of the even order are of the magnetic type. It follows from (5.5) that $\vec{r}\vec{H}_{2k} = 0$ and $\vec{r}\vec{E}_{2k} \neq 0$. Correspondingly, radiation fields of the time-dependent currents flowing in a toroidal coil and in toroidal emitters of the odd order are of the electric type.

5.4. Historical remarks to section 5

EMFs (5.4) and (5.6) of elementary toroidal sources were obtained in [39]. Their interactions with an external EMF are given here for the first time.

6. The Lorentz and Feld–Tai lemmas

6.1. Standard derivation of the Lorentz lemma

We write out Maxwell's equations for two current sources $\vec{j}_1$ and $\vec{j}_2$:

$$\mathrm{curl}\,\vec{E}_1 = -\frac{1}{c}\dot{\vec{H}}_1 \qquad \mathrm{curl}\,\vec{H}_1 = \frac{1}{c}\dot{\vec{E}}_1 + \frac{4\pi}{c}\vec{j}_1$$
$$\mathrm{curl}\,\vec{E}_2 = -\frac{1}{c}\dot{\vec{H}}_2 \qquad \mathrm{curl}\,\vec{H}_2 = \frac{1}{c}\dot{\vec{E}}_2 + \frac{4\pi}{c}\vec{j}_2. \qquad (6.1)$$

From this one easily obtains

$$\mathrm{div}(\vec{E}_1\times\vec{H}_2) = \vec{H}_2\,\mathrm{curl}\,\vec{E}_1 - \vec{E}_1\,\mathrm{curl}\,\vec{E}_2$$
$$= -\frac{1}{c}\vec{H}_2\dot{\vec{H}}_1 - \frac{1}{c}\vec{E}_1\dot{\vec{E}}_2 - \frac{4\pi}{c}\vec{j}_2\vec{E}_1$$
$$\mathrm{div}(\vec{E}_2\times\vec{H}_1) = \vec{H}_2\,\mathrm{curl}\,\vec{E}_1 - \vec{E}_1\,\mathrm{curl}\,\vec{E}_2$$
$$= -\frac{1}{c}\vec{H}_1\dot{\vec{H}}_2 - \frac{1}{c}\vec{E}_2\dot{\vec{E}}_1 - \frac{4\pi}{c}\vec{j}_1\vec{E}_2.$$

Subtracting these equations from each other, one obtains

$$\mathrm{div}(\vec{E}_1\times\vec{H}_2 - \vec{E}_2\times\vec{H}_1) = \frac{1}{c}(\vec{H}_1\dot{\vec{H}}_2 - \vec{H}_2\dot{\vec{H}}_1)$$
$$-\frac{1}{c}(\vec{E}_1\dot{\vec{E}}_2 - \vec{E}_2\dot{\vec{E}}_1) + \frac{4\pi}{c}(\vec{j}_1\vec{E}_2 - \vec{j}_1\vec{E}_2). \qquad (6.2)$$

When the time dependence of the field strengths is given by $\exp(\mathrm{i}\omega t)$, i.e.

$$\vec{E}_1 = \exp(\mathrm{i}\omega t)\vec{E}_1^0 \qquad \vec{E}_2 = \exp(\mathrm{i}\omega t)\vec{E}_2^0$$
$$\vec{H}_1 = \exp(\mathrm{i}\omega t)\vec{H}_1^0 \qquad \vec{H}_2 = \exp(\mathrm{i}\omega t)\vec{H}_2^0 \qquad (6.3)$$

then

$$\vec{E}_1\dot{\vec{E}}_2 = \vec{E}_2\dot{\vec{E}}_1 \qquad \vec{H}_1\dot{\vec{H}}_2 = \vec{H}_2\dot{\vec{H}}_1 \qquad (6.4)$$

and

$$\mathrm{div}(\vec{E}_1\times\vec{H}_2 - \vec{E}_2\times\vec{H}_1) = \frac{4\pi}{c}(\vec{j}_1\vec{E}_2 - \vec{j}_1\vec{E}_2).$$

Integrate this relation over the sphere of the radius R_0 and apply the Gauss theorem

$$R_0^2\int(\vec{E}_1\times\vec{H}_2 - \vec{E}_2\times\vec{H}_1)_r\,\mathrm{d}\Omega = \frac{4\pi}{c}\int(\vec{j}_1\vec{E}_2 - \vec{j}_1\vec{E}_2)\,\mathrm{d}V. \qquad (6.5)$$

For $R_0 \to \infty$, the left-hand side (LHS) of this equation disappears and one obtains the famous Lorentz lemma

$$\mathcal{E}_{12} = \mathcal{E}_{21} \qquad (6.6)$$

where we put $\mathcal{E}_{12} = \int \vec{j}_1\vec{E}_2\,\mathrm{d}V$ and $\mathcal{E}_{21} = \int \vec{j}_2\vec{E}_1\,\mathrm{d}V$.

6.2. The Feld–Tai lemma

The Feld–Tai lemma states that

$$\mathcal{H}_{12} = \mathcal{H}_{21} \qquad (6.7)$$

where $\mathcal{H}_{12} = \int \vec{j}_1\vec{H}_2\,\mathrm{d}V$ and $\mathcal{H}_{21} = \int \vec{j}_2\vec{H}_1\,\mathrm{d}V$. It is proved along the same lines as the Lorentz lemma. From (6.1) one easily obtains

$$\mathrm{div}(\vec{H}_1\times\vec{H}_2) = \frac{1}{c}(\vec{H}_2\dot{\vec{E}}_1 - \vec{H}_1\dot{\vec{E}}_2) + \vec{j}_1\vec{H}_2 - \vec{j}_1\vec{H}_2$$

$$\mathrm{div}(\vec{E}_1 \times \vec{E}_2) = \frac{1}{c}(\vec{E}_1 \dot{\vec{H}}_2 - \vec{E}_2 \dot{\vec{H}}_1). \tag{6.8}$$

Subtracting these equations from each other, one obtains

$$\mathrm{div}(\vec{E}_1 \times \vec{E}_2 - \vec{H}_1 \times \vec{H}_2) = \frac{1}{c}(\vec{E}_1 \dot{\vec{H}}_2 - \vec{E}_2 \dot{\vec{H}}_1) - \frac{1}{c}(\vec{H}_2 \dot{\vec{E}}_1 - \vec{H}_1 \dot{\vec{E}}_2) - \vec{j}_1 \vec{H}_2 + \vec{j}_2 \vec{H}_1. \tag{6.9}$$

If the time dependences of $\vec{E}$ and $\vec{H}$ are $\exp(\mathrm{i}\omega t)$, then the first two terms in the right-hand side (RHS) of (6.9) cancel each other. Integrating the remaining ones over the whole volume, one obtains

$$r^2 \int \mathrm{d}\Omega\, (\vec{E}_1 \times \vec{E}_2 - \vec{H}_1 \times \vec{H}_2)_r = \int \mathrm{d}V\, (-\vec{j}_1 \vec{H}_2 + \vec{j}_2 \vec{H}_1). \tag{6.10}$$

Since $\vec{E} = \vec{H} \times \vec{n}$ and $\vec{n} = \vec{r}/r$ on the sphere of infinite radius, the LHS of (6.10) disappears and one obtains the Feld–Tai lemma (6.7).

6.3. Lorentz and Feld–Tai lemmas for real time dependences

The crucial point in obtaining (6.6) and (6.7) is (6.3). However, the real current densities should be real. The possibility of operating with complex quantities like

$$\exp(\mathrm{i}\omega t)\vec{j} \qquad \exp(\mathrm{i}\omega t)\vec{E} \qquad \exp(\mathrm{i}\omega t)\vec{H}$$

is valid as far as we deal with the quantities linear in field strengths. For example, if the actual dependence of the current density is $\cos\omega t$, then we may solve Maxwell's equations with $\exp(\mathrm{i}\omega t)\vec{j}$, $\exp(\mathrm{i}\omega t)\vec{E}$ and $\exp(\mathrm{i}\omega t)\vec{H}$ and at the end of calculations take the real parts of these quantities. However, one should be very careful in dealing with quadratic combinations such as (6.2) and (6.9). To avoid mistakes one should first take real parts of the EMF strengths and substitute them into quadratic combinations of the field strengths. Consider the two equalities, (6.4), obtained under assumption (6.3). Equations (3.9), (3.23) and (4.4) show that actual field strengths contain both $\cos\omega t$ and $\sin\omega t$

$$\vec{E}_1 = \cos\omega t \vec{E}_1^c + \sin\omega t \vec{E}_1^s \qquad \vec{E}_2 = \cos\omega t \vec{E}_2^c + \sin\omega t \vec{E}_2^s$$
$$\vec{H}_1 = \cos\omega t \vec{H}_1^c + \sin\omega t \vec{H}_1^s \qquad \vec{H}_2 = \cos\omega t \vec{H}_2^c + \sin\omega t \vec{H}_2^s. \tag{6.11}$$

Substituting (6.11) into (6.4), we find that (6.4) are satisfied if

$$\vec{E}_1^c \vec{E}_2^s = \vec{E}_1^s \vec{E}_2^c \qquad \vec{H}_1^c \vec{H}_2^s = \vec{H}_1^s \vec{H}_2^c. \tag{6.12}$$

It is not evident that these equations are fulfilled for the real time dependences, $\cos\omega t$ and $\sin\omega t$. We show below (section 6.5) that they are not satisfied even for the simplest of EMF sources.

6.4. The Lorentz and Feld–Tai lemmas for elementary electromagnetic sources

We apply now the Lorentz and Feld–Tai lemmas to the simplest electromagnetic sources. The general conditions for the validity and violation of these lemmas will be given in section 7.3. The verification of the Lorentz lemma validity for particular sources is needed because the RHS and LHS of this lemma are the experimentally observed voltages (see section 7.3.3 for details) induced in these particular sources. Their deviation from the theoretical values testify to the possible violation of reciprocity (see section 7.4).

6.4.1. Interacting electric dipole and current loop. Equations (3.16) and (3.17) define the current density and EMF strengths of the current loop, respectively. Correspondingly, equations (4.1) and (4.2) define the same quantities for the electric dipole. Combining them, we evaluate the integrals entering into the Lorentz and Feld–Tai lemmas:

$$\begin{aligned}
\mathcal{E}_{Ld} &= f_L \int \mathrm{curl}(\vec{n}_L \delta^3(\vec{r} - \vec{r}_L))\vec{E}_d\, \mathrm{d}V \\
&= -\frac{1}{c} f_L \vec{n}_L \int \delta^3(\vec{r} - \vec{r}_L)\dot{\vec{H}}_d\, \mathrm{d}V \\
&= \frac{1}{c^3 R_{dL}^2} f_L(t)(\vec{R}_{Ld}(\vec{n}_d \times \vec{n}_L))\ddot{D}_d \\
\mathcal{E}_{dL} &= \frac{1}{c^3 R_{dL}^2} \dot{f}_d(t)(\vec{R}_{dL}(\vec{n}_L \times \vec{n}_d))\dot{D}_L \\
\mathcal{H}_{Ld} &= -\frac{1}{c^3 R_{dL}} f_L(t) \\
&\quad \times \left[(\vec{n}_d \vec{n}_L)\dot{G}_d - \frac{1}{R_{dL}^2}(\vec{n}_d \vec{R}_{dL})(\vec{n}_L \vec{R}_{dL})\dot{F}_d\right] \\
\mathcal{H}_{dL} &= -\frac{1}{c^3 R_{dL}} \dot{f}_d(t) \\
&\quad \times \left[(\vec{n}_d \vec{n}_L)G_L - \frac{1}{R_{dL}^2}(\vec{n}_d \vec{R}_{dL})(\vec{n}_L \vec{R}_{dL})F_L\right].
\end{aligned} \tag{6.13}$$

Here $\vec{R}_{Ld} = -\vec{R}_{dL} = \vec{r}_L - \vec{r}_d$, $D_L = D(f_L)$, $D_d = D(f_d)$, etc. The functions D, F and G are defined by (3.17). The argument of f functions entering into D, F and G is $t - R_{dL}/c$. We see that

$$\mathcal{E}_{Ld} = \mathcal{E}_{dL} \qquad \text{and} \qquad \mathcal{H}_{Ld} = \mathcal{H}_{dL} \tag{6.14}$$

for arbitrary $f_L = \dot{f}_d$.

6.4.2. Interacting electric dipole and TS. Combining equations (3.46) and (3.47) that define the current density and EMF strengths of a TS and equations (4.1) and (4.2) that define the same quantities for the electric dipole, we evaluate the integrals entering into the Lorentz and Feld–Tai lemmas:

$$\begin{aligned}
\mathcal{E}_{Td} &= f_T(t) \int \mathrm{curl}^{(2)}(\vec{n}_T \delta^3(\vec{r} - \vec{r}_T)\vec{E}_d\, \mathrm{d}V \\
&= -\frac{1}{c^2} f_T(t)\ddot{\vec{E}}_d(\vec{R}_{TD}) \\
&= \frac{f_T(t)}{c^4 R_{dT}} \left[(\vec{n}_T \vec{n}_d)\ddot{G}_d - \frac{1}{R_{dT}^2}(\vec{n}_d \vec{R}_{Td})(\vec{n}_T \vec{R}_{Td})\ddot{F}_d\right] \\
\mathcal{E}_{dT} &= \dot{f}_d(t)\frac{1}{c^4 R_{dT}} \\
&\quad \times \left[(\vec{n}_T \vec{n}_d)\dot{G}_T - \frac{1}{R_{dT}^2}(\vec{n}_d \vec{R}_{Td})(\vec{n}_T \vec{R}_{Td})\dot{F}_T\right] \\
\mathcal{H}_{Td} &= -\frac{1}{c^2} f_T(t)(\vec{n}_T \ddot{\vec{H}}_d(\vec{R}_{TD}) \\
&= -\frac{1}{c^4 R_{Td}^2} f_T \vec{R}_{Td}(\vec{n}_d \times \vec{n}_T)D_d^{(3)} \\
\mathcal{H}_{dT} &= \frac{1}{c^4 R_{Td}^2} \dot{f}_d(t)\vec{R}_{dT}(\vec{n}_d \times \vec{n}_T)D_T^{(2)}.
\end{aligned} \tag{6.15}$$

The dots above the field strengths denote time derivatives. Again, we see that these integrals coincide for arbitrary $f_T = \dot{f}_d$.

6.4.3. Interacting current loop and TS. Finally, using equations (3.16), (3,17), (3.46) and (3.47) we obtain for the integrals entering into the Lorentz and Feld–Tai lemmas:

$$\begin{aligned}
\mathcal{E}_{LT} &= f_L \int \operatorname{curl}(\vec{n}_L \delta^3(\vec{r}-\vec{r}_L))\vec{E}_T(\vec{r}-\vec{r}_T)\,\mathrm{d}V \\
&= -\frac{1}{c} f_L \vec{n}_L \int \delta^3(\vec{r}-\vec{r}_L)\dot{\vec{H}}_T(\vec{r}-\vec{r}_T)\,\mathrm{d}V \\
&= -\frac{1}{c^5 R_{LT}^2} f_L \vec{R}_{LT}(\vec{n}_T \times \vec{n}_L) D_T^{(3)} \\
\mathcal{E}_{TL} &= f_T(t) \int \operatorname{curl}^{(2)}(\vec{n}_T \delta^3(\vec{r}-\vec{r}_T))\vec{E}_L(\vec{r}-\vec{r}_L)\,\mathrm{d}V \\
&= -f_T(t)\frac{1}{c^2}\vec{n}_T \ddot{\vec{E}}_L(\vec{R}_{TL}) \\
&= -\frac{1}{c^5 R_{TL}^2} f_T(t) D_L^{(3)} \vec{R}_{TL}(\vec{n}_L \times \vec{n}_T) \\
\mathcal{H}_{LT} &= f_L \int \operatorname{curl}(\vec{n}_L \delta^3(\vec{r}-\vec{r}_L))\vec{H}_T(\vec{r}-\vec{r}_T)\,\mathrm{d}V \\
&= \frac{1}{c} f_l \vec{n}_L \dot{\vec{E}}_T(\vec{R}_{LT}) \\
&= \frac{1}{c^5 R_{LT}} f_L \left[(\vec{n}_L\vec{n}_T)\ddot{G}_T - \frac{1}{R_{dT}^2}(\vec{n}_L\vec{R}_{LT})(\vec{n}_T\vec{R}_{LT})\ddot{F}_T\right] \\
\mathcal{H}_{TL} &= f_T \int \operatorname{curl}^{(2)}(\vec{n}_T \delta^3(\vec{r}-\vec{r}_T))\vec{H}_L\,\mathrm{d}V \\
&= -f_T\frac{1}{c^2}\vec{n}_T \int \delta^3(\vec{r}-\vec{r}_T)\ddot{\vec{H}}_L(\vec{r}-\vec{R}_L)\,\mathrm{d}V \\
&= -\frac{f_T}{c^2}\vec{n}_T \ddot{\vec{H}}(\vec{R}_{TL}) \\
&= \frac{f_T}{c^5 R_{TL}}\left[(\vec{n}_L\vec{n}_T)\ddot{G}_L - \frac{1}{R_{dT}^2}(\vec{n}_L\vec{R}_{LT})(\vec{n}_T\vec{R}_{LT})\ddot{F}_L\right].
\end{aligned} \tag{6.16}$$

We see that these integrals coincide for arbitrary $f_T = f_L$.

6.5. The Lorentz and Feld–Tai lemmas may be fulfilled even when condition (6.4), ensuring their validity, is violated

We analyse the conditions (6.4) and (6.12) using the interacting current loop and TS as an example. As we have seen, the equalities

$$\mathcal{E}_{LT} = \mathcal{E}_{TL} \qquad \text{and} \qquad \mathcal{H}_{LT} = \mathcal{H}_{TL}$$

are satisfied if $f_T = f_L$. However, it is easy to check that the conditions, (6.4) and (6.12), under which the Lorentz and Feld–Tai lemmas were obtained are not satisfied for arbitrary $f_T = f_L$. More accurately, (6.4) and (6.12) are valid if the time dependences f_T and f_L are of the following specific form: $f_T \sim \exp(\mathrm{i}\omega t)$ and $f_L \sim \exp(\mathrm{i}\omega t)$. But how to reconcile the violation of (6.4) and (6.12) with the fulfillment of (6.6) and (6.7) proved in a previous section? The answer is that although the left- and right-hand sides of (6.4) do not coincide for interacting current loop and TS with arbitrary time dependence, space integrals from both sides of (6.4) do coincide. This, in turn, means that the Lorentz and Feld–Tai lemmas have a greater range of applicability than suggested up to now. The same conclusions are valid for the interaction of an electric dipole with the current loop and with the TS. The fact that the Lorentz lemma (6.6) may be fulfilled due to the equalities of the space integrals from (6.4), not to (6.4) itself, was earlier recognized by Ginzburg [43].

6.6. Historical remarks to section 6

The standard derivation of the Lorentz lemma may be found in many textbooks (see, e.g., [44–46]). The derivation of the Feld–Tai lemma is available only in journal papers [11–13]. The mutual interaction of a point-like electric dipole, current loop and TS is given here for the first time.

7. Alternative proof of the Lorentz and Feld–Tai lemmas

7.1. Digression on the energy exchange

At first we consider a simpler case, corresponding to the energy exchange between two sources of electromagnetic energy. The energy transmitted from one charge–current source $\rho_2(\vec{r},t)$, $\vec{j}_2(\vec{r},t)$ to the other source $\rho_1(\vec{r},t)$, $\vec{j}_1(\vec{r},t)$ is given by

$$W_{12}(t) = \int \left[\rho_1(\vec{r}_1,t)\Phi_2(\vec{r}_1,t) - \frac{1}{c}\vec{j}_1(\vec{r}_1,t)\vec{A}_2(\vec{r}_1,t)\right]\mathrm{d}V_1, \tag{7.1}$$

where $\Phi_2(\vec{r}_1,t)$ and $\vec{A}_2(\vec{r}_1,t)$ are the scalar and electric potentials induced by the charge–current density $(\rho_2, \vec{j}_2)$ at the position of the charge–current density $(\rho_1, \vec{j}_1)$. They are given by

$$\begin{aligned}
\Phi_2(\vec{r}_1,t) &= \int \frac{1}{R_{12}}\rho_2(\vec{r}_2,\tau)\delta(\tau - t + R_{12}/c)\,\mathrm{d}V_2\,\mathrm{d}\tau \\
\vec{A}_2(\vec{r}_1,t) &= \frac{1}{c}\int \frac{1}{R_{12}}\vec{j}_2(\vec{r}_2,\tau)\delta(\tau - t + R_{12}/c)\,\mathrm{d}V_2\,\mathrm{d}\tau.
\end{aligned} \tag{7.2}$$

Here $R_{12} = |\vec{r}_1 - \vec{r}_2|$ is the distance between the particular point of sources 1 and 2. Substituting this into (7.1), one obtains

$$W_{12}(t) = \int \left[\rho_1(\vec{r}_1,t)\rho_2(\vec{r}_2,\tau) - \frac{1}{c^2}\vec{j}_1(\vec{r}_1,t)\vec{j}_2(\vec{r}_2,\tau)\right] \times \frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)\,\mathrm{d}V_1\,\mathrm{d}V_2\,\mathrm{d}\tau. \tag{7.3}$$

In the same way,

$$W_{21}(t) = \int \left[\rho_2(\vec{r}_2,t)\rho_1(\vec{r}_1,\tau) - \frac{1}{c^2}\vec{j}_1(\vec{r}_1,\tau)\vec{j}_2(\vec{r}_2,t)\right] \times \frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)\,\mathrm{d}V_1\,\mathrm{d}V_2\,\mathrm{d}\tau. \tag{7.4}$$

We see, that in general $W_{21}(t) \neq W_{12}(t)$. Let now the time dependences in ρ and $\vec{j}$ be separated

$$\begin{aligned}
\rho_1(\vec{r}_1,t_1) &= \rho_1(t_1)\rho_1(\vec{r}_1) \qquad & \rho_2(\vec{r}_2,t_2) &= \rho_2(t_2)\rho_2(\vec{r}_2) \\
\vec{j}_1(\vec{r}_1,t_1) &= j_1(t_1)\vec{j}_1(\vec{r}_1) \qquad & \vec{j}_2(\vec{r}_2,t_2) &= j_2(t_2)\vec{j}_2(\vec{r}_2).
\end{aligned} \tag{7.5}$$

Then,

$$W_{12}(t) = \int \left[\rho_1(t)\rho_1(\vec{r}_1)\rho_2(\vec{r}_2)\rho_2(\tau) - \frac{1}{c^2}j_1(t)\vec{j}_1(\vec{r}_1)\vec{j}_2(\vec{r}_2)j_2(\tau)\right] \times \frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)\,\mathrm{d}V_1\,\mathrm{d}V_2\,\mathrm{d}\tau \tag{7.6}$$

$$W_{21}(t) = \int \left[\rho_2(t)\rho_2(\vec{r}_2)\rho_1(\vec{r}_1)\rho_1(\tau) - \frac{1}{c^2}j_2(t)\vec{j}_2(\vec{r}_2)\vec{j}_1(\vec{r}_1)j_1(\tau)\right] \times \frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)\,\mathrm{d}V_1\,\mathrm{d}V_2\,\mathrm{d}\tau. \tag{7.7}$$

It follows from this that $W_{12} = W_{21}$ if the time dependences of sources 1 and 2 coincide, i.e. when

$$\rho_1(t) = \rho_2(t) \qquad j_1(t) = j_2(t). \tag{7.8}$$

That is, the action and reaction coincide if the time dependences of sources 1 and 2 are separated and synchronized.

The violation of action and reaction due to the retarded nature of electromagnetic interaction was first recognized by Lorentz in 1895 [47]. As far as we know, the best exposition of these questions was given in Cullwick's book [48] where the explicit violation of action and reaction equality was demonstrated for the interaction of a charge with a TS. In a modern physical literature the violation of this equality is considered as almost obvious. We quote, for example, French [49]:

> The equality of action and reaction has almost no place in relativistic mechanics. It must be essentially a statement about the forces acting on two bodies, as a result of their mutual interaction at a given instant. And, because of the relativity of simultaneity, this phrase has no meaning.

The violation of action and reaction equality for the interaction between the moving current loop and charge and between two moving charges was noted by Jefimenko [50] and Cornille [51], respectively. However, this violation is not restricted only to the retardation effects. Even for the interacting static metallic currents there are two known interaction laws: Ampere's law which agrees with Newton's third law (equality of action and reaction forces) and Lorentz's law which violates it (see, e.g., [52, 53]). However, if the above currents are closed, the difference between these forces disappears: both of them satisfy Newton's third law [54] (from our considerations it follows that in the non-static case the violation of the action–reaction equality is possible even for closed coils). As to experiments, some of them [55] support only Ampere's law of force, while others [56] give the same result for both laws. These questions are beyond the present consideration.

7.2. Concrete examples: the energy exchange between elementary toroidal sources

Let us have two toroidal sources TS_1 and TS_2 in an arbitrary order. Their interaction energy is

$$W = W_{12} + W_{21}$$

where W_{12} and W_{21} are the parts of W localized at the positions of TS_1 and TS_2. More accurately, W_{12} is the energy induced by source 2 at the position of source 1; similarly for W_{21}. They are given by

$$W_{12} = -\frac{1}{c}\int \vec{j}_1(\vec{r}-\vec{r}_1)\vec{A}_2(\vec{r}-\vec{r}_2)\,\mathrm{d}V \tag{7.9}$$

and

$$W_{21} = -\frac{1}{c}\int \vec{j}_2(\vec{r}-\vec{r}_2)\vec{A}_1(\vec{r}-\vec{r}_1)\,\mathrm{d}V \tag{7.10}$$

respectively.

7.2.1. The interaction of even toroidal sources. Let $\vec{j}_1$ and $\vec{j}_2$ be both of even order

$$\vec{j}_1 = f_1(t)\,\mathrm{curl}^{2l_1+1}[\vec{n}_1\delta^3(\vec{r}-\vec{r}_1)]$$
$$\vec{j}_2 = f_2(t)\,\mathrm{curl}^{2l_2+1}[\vec{n}_2\delta^3(\vec{r}-\vec{r}_2)].$$

Then,

$$W_{12} = \frac{(-1)^{l_1+1}}{c^{2l_1+1}} f_1(t)\vec{n}_1\cdot\vec{H}_2^{(2l_1)}(\vec{R}_{12})$$
$$W_{21} = \frac{(-1)^{l_2+1}}{c^{2l_2+1}} f_2(t)\vec{n}_2\cdot\vec{H}_1^{(2l_2)}(\vec{R}_{21}) \tag{7.11}$$

where $\vec{H}_2^{(2l_1)}(\vec{R}_{12})$ is the $2l_1$ time derivative of the magnetic field produced by TS_2 at the position of TS_1 and $\vec{H}_1^{(2l_2)}(\vec{R}_{21})$ is the $2l_2$ time derivative of the magnetic field produced by TS_1 at the position of TS_2. Substituting them from (5.4), one obtains

$$W_{12} = f_1\frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+4}R_{12}} \times\left[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12})F_2^{(2l_1+2l_2)} - (\vec{n}_1\vec{n}_2)G_2^{(2l_1+2l_2)}\right]$$
$$W_{21} = f_2\frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+4}R_{12}} \times\left[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12})F_1^{(2l_1+2l_2)} - (\vec{n}_1\vec{n}_2)G_1^{(2l_1+2l_2)}\right] \tag{7.12}$$

(the upper indices at F and G functions denote the time derivatives). We see that $W_{12} = W_{21}$ for arbitrary $f_1 = f_2$. Let f_1 and f_2 not depend on time. Then, W_{12} and W_{21} differ from zero only for $l_1 = l_2 = 0$:

$$W_{12} = W_{21} = -\frac{f_1 f_2}{c^2 R_{12}^3}\left[3\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12}) - (\vec{n}_1\vec{n}_2)\right]$$

which coincides with interaction of two magnetic dipoles.

7.2.2. The interaction of odd toroidal sources. Let $\vec{j}_1$ and $\vec{j}_2$ both be of odd order.

$$\vec{j}_1 = f_{2l_1+1}(t)\,\mathrm{curl}^{2l_1+2}[\vec{n}_1\delta^3(\vec{r}-\vec{r}_1)]$$
$$\vec{j}_2 = f_{2l_2+1}(t)\,\mathrm{curl}^{2l_2+2}[\vec{n}_2\delta^3(\vec{r}-\vec{r}_2)].$$

Then

$$W_{12} = \frac{(-1)^{l_1+1}}{c^{2l_1+1}} f_1(t)\vec{n}_1\cdot\vec{E}_2^{(2l_1+1)}(\vec{R}_{12})$$
$$W_{21} = \frac{(-1)^{l_2+1}}{c^{2l_2+1}} f_2(t)\vec{n}_2\cdot\vec{E}_1^{(2l_2+1)}(\vec{R}_{21}) \tag{7.13}$$

where $\vec{E}_2^{(2l_1+1)}(\vec{R}_{12})$ is the $2l_1+1$ time derivative of the electric field induced by TS_2 at the position of TS_1 and $\vec{E}_1^{(2l_2+1)}(\vec{R}_{21})$ is the $2l_2+1$ time derivative of the electric field induced by TS_1 at the position of TS_2. Substituting them from (5.5), we obtain

$$W_{12} = f_1\frac{(-1)^{l_1+l_2}}{c^{2l_1+2l_2+6}R_{12}} \times\left[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12})F_2^{(2l_1+2l_2+2)} - (\vec{n}_1\vec{n}_2)G_2^{(2l_1+2l_2+2)}\right]$$

$$W_{21} = f_2 \frac{(-1)^{l_1+l_2}}{c^{2l_1+2l_2+6} R_{12}} \times \left[\frac{1}{R_{12}^2} (\vec{n}_1 \vec{R}_{12})(\vec{n}_2 \vec{R}_{12}) F_1^{(2l_1+2l_2+2)} - (\vec{n}_1 \vec{n}_2) G_1^{(2l_1+2l_2+2)} \right]. \quad (7.14)$$

Again, we see that $W_{12} = W_{21}$ for arbitrary $f_1 = f_2$. Let f_1 and f_2 not depend on time. Then, $W_{12} = W_{21} = 0$. This means that static toroidal sources of an odd order (and, in particular, usual static TSs) do not interact.

It follows from (7.12) and (7.14) that the interaction energy between toroidal sources of the same order vanishes when the two following conditions are fulfilled simultaneously.

(i) The symmetry axes of toroidal sources are mutually orthogonal.
(ii) The symmetry axes of toroidal sources are perpendicular to the vector $\vec{R}_{12}$ going from TS_1 to, TS_2.

In particular, this is valid for two interacting current loops or TSs.

7.2.3. The interaction of even and odd toroidal sources. Let one of the currents be of the even order and the other of the odd one:

$$\vec{j}_1 = f_1(t)\,\mathrm{curl}^{2l_1+1}[\vec{n}_1 \delta^3(\vec{r} - \vec{r}_1)]$$
$$\vec{j}_2 = f_2(t)\,\mathrm{curl}^{2l_2+2}[\vec{n}_2 \delta^3(\vec{r} - \vec{r}_2)].$$

Then,

$$W_{12} = \frac{(-1)^{l_1+1}}{c^{2l_1+1}} f_1(t) \vec{n}_1 \cdot \vec{H}_2^{(2l_1)}(\vec{R}_{12})$$
$$W_{21} = \frac{(-1)^{l_2+1}}{c^{2l_2+2}} f_2(t) \vec{n}_2 \cdot \vec{E}_1^{(2l_2+1)}(\vec{R}_{21}). \quad (7.15)$$

We observe a curious fact: TS_1 interacts with the time derivatives of the magnetic field induced by TS_2 while TS_2 interacts with the time derivatives of the electric field induced by TS_1 (by ‘interacts with time derivative’ we mean that the time derivative of the corresponding order enters into the interaction energy). Substitution of $\vec{E}_1$ from (5.4) and $\vec{H}_2$ from (5.5) gives

$$W_{12} = f_1 \frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+5}} \frac{1}{R_{12}^2} \vec{n}_1 (\vec{R}_{12} \times n_2) D_2^{(2l_1+2l_2+2)}$$
$$W_{21} = f_2 \frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+5}} \frac{1}{R_{12}^2} \vec{n}_2 (\vec{R}_{21} \times n_1) D_1^{(2l_1+2l_2+2)}. \quad (7.16)$$

Again, we observe that $W_{12} = W_{21}$ for arbitrary $f_1 = f_2$. From this one can see at once the violation of the action–reaction equality for $f_1 \neq f_2$. Take, for example, the last equation. Let f_1 and f_2 depend and not depend on time, respectively. Then, $W_{12} = 0$ and $W_{21} \neq 0$. This means that TS_1 acts on TS_2 while TS_2 does not act on TS_1. It follows from (7.16) that the interaction energy between toroidal sources of even and odd orders is zero if one of the following two conditions is fulfilled.

(i) When the symmetry axes of TS_1 and TS_2 are parallel.
(ii) When at least one of the two symmetry axes (TS_1 or TS_2) is parallel to the vector $\vec{R}_{12}$ going from TS_1 to TS_2.

In particular, this is valid for the interaction of a current loop with a TS.

7.2.4. Numerical estimations. To explicitly see at what level the equality of action and reaction is violated, consider an interacting current loop and TS with a constant current in its winding. Since, there is no EMF outside such a TS it does not act on the current loop. On the other hand, the action of the current loop on the TS is given by (7.16) where one should put $l_1 = l_2 = 0$. Then,

$$W_{LT} = 0 \qquad W_{TL} = -f_T \frac{1}{c^5 R_{TL}^2} \vec{R}_{TL} (\vec{n}_L \times \vec{n}_T) D_L^{(2)}.$$

Let f_L periodically change with time. Then,

$$f_L = \pi I_L d_L^2 \cos \omega t$$
$$D_L = -\pi I_L d_L^2 \omega \left(\sin \omega t_r - \frac{1}{kr} \cos \omega t_r \right)$$
$$D_L^{(2)} = -\omega^2 D_L \qquad k = \frac{\omega}{c} \qquad t_r = t - \frac{R_{TL}}{c}$$
$$W_{TL} = -f_T \frac{\pi I_L d_L^2}{c^5 R_{TL}^2} \vec{R}_{TL} (\vec{n}_L \times \vec{n}_T) D_L^{(2)} \omega^3 \times \left(\sin \omega t_r - \frac{1}{kr} \cos \omega t_r \right).$$

Now we choose f_T. It is equal to $\pi c g d_T R^2/4$, where $g = 2NI_T/c$, N is a number of coils in a TS winding and I_T is a current in a particular coil. However, instead of a TS winding, it is convenient to use a ferromagnetic ring magnetized in the azimuthal direction (see the middle part of figure 3). These two objects are completely equivalent as to their interaction with an external EMF. The magnetic field inside a TS is given by $H_\phi = g/\rho$, where ρ is the cylindrical radius. If the major radius d_T of a TS is much larger than its minor radius R, we may put $H_\phi = H_T = g/d_T$ and $g = d_T H_T$. Finally, for W_{TL}, we obtain

$$W_{TL} = -\frac{\pi^2 I_L H_T d_L^2 d_T^2 R^2}{c^5 R_{TL}^2} \vec{R}_{TL} (\vec{n}_L \times \vec{n}_T) D_L^{(2)} \omega^3 \times \left(\sin \omega t_r - \frac{1}{kr} \cos \omega t_r \right).$$

Its maximal absolute value is

$$|W_{TL}| = \pi^2 I_L H_T d_L^2 d_T^2 R^2 \omega^3 / c^5 R_{TL}.$$

This expression should be multiplied by the number, N_L, of the turns in a circular loop. The typical value of magnetic field inside the ferromagnetic sample is about 1000 G. Let $N_L = 1000$, $I_L = 1$ A, the dimensions of a current loop and TS are of the order of few centimetres and the distance between sources about 10 cm. In order that the motion of the TS can be observed, the frequency should be of the order few hertz (otherwise, positive and negative values of W_{TL} compensate each other for the finite observation time). For these parameters, $W_{TL} \sim 10^{-32}$ ergs and the corresponding force $F_{TL} \sim W_{TL}/R_{TL} \sim 10^{-33}$ dynes. Such a small force could be hardly observed experimentally for the realistic cosine or sine current dependences. Under the influence of a force from a current loop, the TS begins to move. The EMF strengths are non-zero outside the TS, when it moves uniformly in a medium [57, 58], or when it is accelerated (both in a medium and under vacuum [36]). The moving TS will act on a current loop which, in turn, begins to move. However, these, next-order effects, are beyond the present consideration.

At first glance it seems that the violation of the action–reaction equality testifies to the energy–momentum non-conservation. Fortunately, this is not so. In fact, the energy–momentum balance restores if one takes into account the energy–momentum carried out by the radiated EMF.

7.3. Back to the Lorentz and Feld–Tai lemmas

7.3.1. Lorentz lemma. Proceeding in the same way as for the interaction energies, we obtain for the integrals $\mathcal{E}_{12}$ and $\mathcal{E}_{21}$ entering into the formulation of the Lorentz lemma

$$\mathcal{E}_{12} = -\int \dot\rho_1(\vec r_1, t)\rho_2(\vec r_2, \tau)\delta(\tau - t + R_{12}/c)\frac{1}{R_{12}}\,dV_1\,dV_2\,d\tau + \frac{1}{c^2}\int \vec j_1(\vec r_1, t)\vec j_2(\vec r_2, \tau)\dot\delta(\tau - t + R_{12}/c)\frac{1}{R_{12}}\,dV_1\,dV_2\,d\tau \tag{7.17}$$

$$\mathcal{E}_{21} = -\int \dot\rho_2(\vec r_2, t)\rho_1(\vec r_1, \tau)\delta(\tau - t + R_{12}/c)\frac{1}{R_{12}}\,dV_1\,dV_2\,d\tau + \frac{1}{c^2}\int \vec j_2(\vec r_2, t)\vec j_1(\vec r_1, \tau)\dot\delta(\tau - t + R_{12}/c)\frac{1}{R_{12}}\,dV_1\,dV_2\,d\tau \tag{7.18}$$

where the dot above ρ denotes the derivative with respect to (w.r.t.) t and the dot above the δ function denotes the derivative w.r.t. its argument. Again we see that, in general, $\mathcal{E}_{12}(t) \neq \mathcal{E}_{21}(t)$. Let now the time dependences of ρ and $\vec j$ be separated in the same way as in (7.5). Then,

$$\mathcal{E}_{12} = -\int \dot\rho_1(t)\rho_1(\vec r_1)\rho_2(\vec r_2)\rho_2(\tau) \times\frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)\,dV_1\,dV_2\,d\tau + \frac{1}{c^2}\int j_1(t)\vec j_1(\vec r_1)\vec j_2(\vec r_2)j_2(\tau) \times\frac{1}{R_{12}}\dot\delta(\tau - t + R_{12}/c)\,dV_1\,dV_2\,d\tau \tag{7.19}$$

$$\mathcal{E}_{21} = -\int \dot\rho_2(t)\rho_2(\vec r_2)\rho_1(\vec r_1)\rho_1(\tau) \times\frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)\,dV_1\,dV_2\,d\tau + \frac{1}{c^2}\int j_2(t)\vec j_2(\vec r_2)\vec j_1(\vec r_1)j_1(\tau) \times\frac{1}{R_{12}}\dot\delta(\tau - t + R_{12}/c)\,dV_1\,dV_2\,d\tau \tag{7.20}$$

Similarly to the interaction energies, we see that $\mathcal{E}_{12}(t) = \mathcal{E}_{21}(t)$ for the arbitrary time dependences ρ_1 and j_1 coinciding with ρ_2 and j_2, i.e. when conditions (7.5) and (7.8) are fulfilled.

7.3.2. Feld–Tai lemma. Direct evaluation of integrals entering into the Feld–Tai lemma gives

$$\mathcal{H}_{12} = \epsilon_{ijk}\int j_{1i}(\vec r_1, t)\frac{\partial A_{2k}}{\partial x_{1j}}\,dV_1 = \epsilon_{ijk}\int j_{1i}(\vec r_1, t)j_{2k}(\vec r_2, \tau)\frac{\partial}{\partial x_{1j}}\frac{1}{R_{12}}\delta\left(\tau - t + \frac{R_{12}}{c}\right)dV_1\,dV_2\,d\tau$$

$$\mathcal{H}_{21} = \epsilon_{ijk}\int j_{2i}(\vec r_2, t)\frac{\partial A_{1k}}{\partial x_{2j}}\,dV_2 = \epsilon_{ijk}\int j_{2i}(\vec r_2, t)j_{1k}(\vec r_1, \tau)\frac{\partial}{\partial x_{2j}}\frac{1}{R_{12}}\delta \times\left(\tau - t + \frac{R_{12}}{c}\right)dV_1\,dV_2\,d\tau. \tag{7.21}$$

Here ϵ_{ijk} is the unit antisymmetrical tensor of the third rank. When the time dependences in current densities are separated ($\vec j(r, t) = j(t)\vec j(r)$), these equations are reduced to

$$\mathcal{H}_{12} = \epsilon_{ijk}j_1(t)\int j_2(\tau)j_{1i}(\vec r_1)j_{2k}(\vec r_2)\frac{\partial}{\partial x_{1j}}\frac{1}{R_{12}}\delta \times\left(\tau - t + \frac{R_{12}}{c}\right)dV_1\,dV_2\,d\tau$$

$$\mathcal{H}_{21} = \epsilon_{ijk}j_2(t)\int j_1(\tau)j_{2i}(\vec r_2)j_{1k}(\vec r_1)\frac{\partial}{\partial x_{2j}}\frac{1}{R_{12}}\delta \times\left(\tau - t + \frac{R_{12}}{c}\right)dV_1\,dV_2\,d\tau. \tag{7.22}$$

Obviously

$$\mathcal{H}_{12} = \mathcal{H}_{21}$$

when the arbitrary time dependences of sources 1 and 2 coincide ($j_1(t) = j_2(t)$).

7.3.3. The physical meaning of the Lorentz and Feld–Tai lemma for the interacting current sources. We conclude: the Lorentz and Feld–Tai lemmas are fulfilled when the two following conditions are satisfied.

(i) Time dependences are separated from space variables in the charge–current densities. This means that the time dependence should be the same for all space points of a particular source.
(ii) The separated time dependence is the same for sources 1 and 2.

The physical meaning of the Lorentz lemma is as follows [9, 59]. The time-dependent magnetic flux penetrating a particular turn of a winding creates an electric field directed along this turn. Being summed, they give the potential difference between the ends of the winding if it is not closed and induce the current in the winding if it is closed. This voltage (or current) can be measured. To obtain voltage, in $\mathcal{E}_{12}$ we omit the time-dependent current force I_1 (not the current density $\vec j_1$). Thus $\mathcal{E}_{12}(t)$ so obtained gives the time-dependent voltage induced in winding 1 by the time-dependent current flowing in winding 2. Similarly, if in $\mathcal{E}_{21}$ we omit the time-dependent current force I_2, then $\mathcal{E}_{21}(t)$ gives the time-dependent voltage induced in winding 2 by the time-dependent current flowing in winding 1. Thus $\mathcal{E}_{12}$ and $\mathcal{E}_{21}$ so obtained coincide if $I_1 = I_2$. We observe that in the first case winding 1 is a receiver and winding 2 is a transmitter. In the second case, the situation is opposite. This means that an induced voltage is invariant under the replacement of the detector and transmitter. We illustrate this using a point-like TS and a current loop as an example. Turning to (3.16) and (3.46), we observe that f_T and f_L in $\mathcal{E}_{TL}$ may be presented as

$$f_T = \frac{\pi N I_T d_T R^2}{2}\tilde f_T \qquad f_L = \pi I_L d_L^2 \tilde f_L$$

where I_T and I_L are the current forces in a TS and current loop, respectively, and $\tilde{f}_T$ and $\tilde{f}_L$ are their time dependences. Omitting the factor $I_T \tilde{f}_T$, for the voltage induced in a TS we obtain

$$V_{TL} = -\frac{\pi^2 N d_T R^2 d_L^2}{2c^5 R_{TL}^2} D^{(3)}(I_L \tilde{f}_L).$$

In the same way, omitting the factor $I_L \tilde{f}_L$, for the voltage induced in a current loop we obtain

$$V_{LT} = -\frac{\pi^2 N d_T R^2 d_L^2}{2c^5 R_{TL}^2} D^{(3)}(I_T \tilde{f}_T).$$

Indeed, we see that $V_{TL} = V_{LT}$ if $I_T \tilde{f}_T = I_L \tilde{f}_L$, i.e. when the time-dependent currents flowing in a current loop and toroidal solenoid are the same.

For completeness, we write out, without derivation, the left-hand ($\mathcal{E}_{12} = \int \mathrm{d}V\, \vec{j}_1(\vec{r} - \vec{r}_1)\vec{E}_2(\vec{r} - \vec{r}_2)\, \mathrm{d}V$) and right-hand ($\mathcal{E}_{21} = \int \mathrm{d}V\, \vec{j}_2(\vec{r} - \vec{r}_2)\vec{E}_1(\vec{r} - \vec{r}_1)\, \mathrm{d}V$) sides, the Lorentz lemma for the toroidal sources, the following.

(i) Both toroidal sources are of even order

$$\mathcal{E}_{12} = f_1 \frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+4} R_{12}} \times \left[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12}) F_2^{(2l_1+2l_2+1)} - (\vec{n}_1\vec{n}_2) G_2^{(2l_1+2l_2+1)}\right]$$

$$\mathcal{E}_{21} = f_2 \frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+4} R_{12}} \times \left[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12}) F_1^{(2l_1+2l_2+1)} - (\vec{n}_1\vec{n}_2) G_1^{(2l_1+2l_2+1)}\right].$$

(ii) Both toroidal sources are of odd order

$$\mathcal{E}_{12} = f_1 \frac{(-1)^{l_1+l_2}}{c^{2l_1+2l_2+6} R_{12}} \times \left[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12}) F_2^{(2l_1+2l_2+3)} - (\vec{n}_1\vec{n}_2) G_2^{(2l_1+2l_2+3)}\right]$$

$$\mathcal{E}_{21} = f_2 \frac{(-1)^{l_1+l_2}}{c^{2l_1+2l_2+6} R_{12}} \times \left[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12}) F_1^{(2l_1+2l_2+3)} - (\vec{n}_1\vec{n}_2) G_1^{(2l_1+2l_2+3)}\right].$$

(iii) One of toroidal sources (source 1) is of even order and the other (source 2) is of odd order

$$\mathcal{E}_{12} = f_1 \frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+5}} \frac{1}{R_{12}^2} \vec{n}_1(\vec{R}_{12} \times n_2) D_2^{(2l_1+2l_2+3)}$$

$$\mathcal{E}_{21} = f_2 \frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+5}} \frac{1}{R_{12}^2} \vec{n}_2(\vec{R}_{21} \times n_1) D_1^{(2l_1+2l_2+3)}.$$

These quantities are proportional to the induced voltages and, thus, have physical meaning. It follows from these equations that the voltages induced in the toroidal sources of the same order vanish when the two following conditions are fulfilled simultaneously:

(i) the symmetry axes of toroidal sources are mutually orthogonal;
(ii) the symmetry axes of toroidal sources are perpendicular to the vector $\vec{R}_{12}$ going from TS_1 to TS_2.

On the other hand, voltages induced in the toroidal sources of opposite orders vanish if one of the two following conditions is fulfilled:

(i) when the symmetry axes of TS_1 and TS_2 are parallel;
(ii) when at least one of two symmetry axes (TS_1 or TS_2) is parallel to the vector $\vec{R}_{12}$ going from TS_1 to TS_2.

These considerations may be useful when planning experiments with reciprocity violation.

The physical meaning of the Feld–Tai lemma for interacting current sources is not clear to us. A time-dependent electric field penetrating a particular turn of a winding creates the magnetic field directed along this turn. If free magnetic charges existed, then integrals entering into the Feld–Tai lemma (after omitting the corresponding factors as in the Lorentz lemma) would give the magnetic voltage between the ends of the winding (if it is not closed). Their equality would give the symmetry between the transmitter and the receiver. Since monopoles have not yet been found, this interpretation of the Feld–Tai lemma has no relation to reality. However, Lakhtakia [14] and Monzon [13], seem to have found numerous applications of the Feld–Tai lemma.

7.3.4. Another viewpoint on the Lorentz and Feld–Tai lemmas. In the Fourier representation ($\vec{E}(t) = \int \vec{E}(\omega)\exp(\mathrm{i}\omega t)\, \mathrm{d}\omega$, etc.) the curl parts of Maxwell equations look like

$$\operatorname{curl} \vec{E} = -\mathrm{i}k\vec{H} \qquad \operatorname{curl} \vec{H} = \mathrm{i}k\vec{E} + \frac{4\pi}{c}\vec{j} \qquad k = \omega/c.$$

Then, the Lorentz and Feld–Tai lemmas are satisfied trivially. For example, the proof of the Lorentz lemma without using the Maxwell equations takes three lines

$$\begin{aligned}
\mathcal{E}_{12} &= \int \vec{j}_1(\vec{r}_1)\vec{E}_{12}(\vec{r}_2)\, \mathrm{d}V_1 \\
&= \int \vec{j}_1(\vec{r}_1)[-\vec{\nabla}\Phi_{12}(\vec{r}_1) - \mathrm{i}k\vec{A}_{12}(\vec{r}_1)]\, \mathrm{d}V_1 \\
&= -\mathrm{i}\omega \int \left[\rho_1(\vec{r}_1)\Phi_{12}(\vec{r}_1) + \frac{1}{c}\vec{j}_1(\vec{r}_1)\vec{A}_{12}(\vec{r}_1)\right] \mathrm{d}V_1 \\
&= -\mathrm{i}\omega \int \left[\rho_1(\vec{r}_1)\rho_2(\vec{r}_2) + \frac{1}{c^2}\vec{j}_1(\vec{r}_1)\vec{j}_2(\vec{r}_2)\right] \\
&\quad \times \frac{\exp(-\mathrm{i}kR_{12})}{R_{12}}\, \mathrm{d}V_1\, \mathrm{d}V_2 \\
&= \mathcal{E}_{21}.
\end{aligned}$$

Therefore, the Lorentz and Feld–Tai lemmas may be viewed as integral relations between the Fourier transforms of the current densities and field strengths. This, in its turn, may be used to derive new identities. For example, multiplying $\mathcal{E}_{12}$ by $\exp(i\omega t)$ and integrating over ω, one gets

$$\begin{aligned}
&\int \vec{j}_1(\vec{r}_1, \omega)\vec{E}_{12}(\vec{r}_1, \omega)\exp(\mathrm{i}\omega t)\, \mathrm{d}V_1\, \mathrm{d}\omega \\
&= \frac{1}{4\pi^2}\int \vec{j}_1(\vec{r}_1, t')\vec{E}_{12}(\vec{r}_1, t'') \\
&\quad \times \exp[\mathrm{i}\omega(t - t' - t'')]\, \mathrm{d}V_1\, \mathrm{d}\omega\, \mathrm{d}t'\, \mathrm{d}t'' \\
&= \frac{1}{2\pi}\int \vec{j}_1(\vec{r}_1, t')\vec{E}_{12}(\vec{r}_1, t'')\delta(t - t' - t'')\, \mathrm{d}V_1\, \mathrm{d}t'\, \mathrm{d}t'' \\
&= \frac{1}{2\pi}\int \vec{j}_1(\vec{r}_1, t - t')\vec{E}_{12}(\vec{r}_1, t')\, \mathrm{d}V_1\, \mathrm{d}t'. \qquad (7.23)
\end{aligned}$$

Performing the same operation with $\mathcal{E}_{21}$ and equalizing the result to (7.23), one arrives at

$$\int \vec{j}_1(\vec{r}_1, t-t')\vec{E}_{12}(\vec{r}_1, t')\,\mathrm{d}V_1\,\mathrm{d}t' = \int \vec{j}_2(\vec{r}_2, t-t')\vec{E}_{21}(\vec{r}_2, t')\,\mathrm{d}V_2\,\mathrm{d}t'. \tag{7.24}$$

This equation was obtained by Feld [60]. We make one further step, excluding electric strengths. Then, the LHS of (7.24) is reduced to

$$\frac{1}{2\pi}\frac{\partial}{\partial t}\int\left[\rho_1(\vec{r}_1, t-t')\rho_2\left(\vec{r}_2, t'-\frac{R_{12}}{c}\right) + \frac{1}{c^2}\vec{j}_1(\vec{r}_1, t-t')\vec{j}_2\left(\vec{r}_2, t'-\frac{R_{12}}{c}\right)\right]\frac{1}{R_{12}}\,\mathrm{d}t'\,\mathrm{d}V_1\,\mathrm{d}V_2.$$

Therefore, the following equation should be satisfied:

$$\begin{aligned}\int&\left[\rho_1(\vec{r}_1, t-t')\rho_2\left(\vec{r}_2, t'-\frac{R_{12}}{c}\right) + \frac{1}{c^2}\vec{j}_1(\vec{r}_1, t-t')\vec{j}_2\left(\vec{r}_2, t'-\frac{R_{12}}{c}\right)\right]\frac{1}{R_{12}}\,\mathrm{d}t'\,\mathrm{d}V_1\,\mathrm{d}V_2\\ &= \int\left[\rho_2(\vec{r}_2, t-t')\rho_1\left(\vec{r}_1, t'-\frac{R_{12}}{c}\right) + \frac{1}{c^2}\vec{j}_2(\vec{r}_2, t-t')\vec{j}_1\left(\vec{r}_1, t'-\frac{R_{12}}{c}\right)\right]\frac{1}{R_{12}}\,\mathrm{d}t'\,\mathrm{d}V_1\,\mathrm{d}V_2.\end{aligned} \tag{7.25}$$

Performing the same operation for the integrals entering into the Feld–Tai lemma, one obtains

$$\begin{aligned}\int &\exp(\mathrm{i}\omega t)\vec{j}_1(\vec{r}_1, \omega)\vec{H}_{12}(\vec{r}_1, \omega)\,\mathrm{d}\omega\,\mathrm{d}V_1\\ &= \frac{1}{2\pi}\int \vec{j}_1(\vec{r}_1, t-t')\vec{H}_{12}(\vec{r}_1, t')\,\mathrm{d}t'\,\mathrm{d}V_1\\ &= -\frac{1}{2\pi c}\int \frac{1}{R_{12}}\,\mathrm{curl}\,\vec{j}_1(\vec{r}_1, t-t') \times\vec{j}_2(\vec{r}_2, t'-R_{12}/c)\,\mathrm{d}V_1\,\mathrm{d}V_2\,\mathrm{d}t'.\end{aligned}$$

Therefore, the following equalities should be fulfilled:

$$\begin{aligned}\int &\vec{j}_1(\vec{r}_1, t-t')\vec{H}_{12}(\vec{r}_1, t')\,\mathrm{d}t'\,\mathrm{d}V_1 = \int \vec{j}_2(\vec{r}_2, t-t')\vec{H}_{21}(\vec{r}_2, t')\,\mathrm{d}t'\,\mathrm{d}V_2\\ \int &\frac{1}{R_{12}}\,\mathrm{curl}\,\vec{j}_1(\vec{r}_1, t-t')\vec{j}_2(\vec{r}_2, t'-R_{12}/c)\,\mathrm{d}V_1\,\mathrm{d}V_2\,\mathrm{d}t'\\ &= \int \frac{1}{R_{12}}\,\mathrm{curl}\,\vec{j}_2(\vec{r}_2, t-t')\vec{j}_1(\vec{r}_1, t'-R_{12}/c)\,\mathrm{d}V_1\,\mathrm{d}V_2\,\mathrm{d}t'.\end{aligned} \tag{7.26}$$

It is important that equations (7.24)–(7.26), contrary to the equations defining the Lorentz and Feld–Tai lemmas, are satisfied for any charge–current density. No assumption on the separation of the space and time dependences or the equality of the time dependences for two interacting sources is needed.

As the author is not the specialist in the applied aspects of reciprocity-like theorems, he cannot appreciate the meaning of the results obtained. On the other hand, there are outstanding experts in this field (A Lakhtakia, J C R Monzon and others). It would be nice to hear their opinion on the treated questions.

7.4. Violation of the Lorentz and Feld–Tai lemmas

Now we analyse the assumption on the separability time and spatial variables in charge–current densities. Take at first the simple circular current loop. Since there are no other turns, there are no resistive or capacity connections between them. Therefore, the current is the same along the whole wire (due to the continuity equation div $\vec{j} = 0$) and the time dependence is clearly separated from the space variables. On the other hand, consider the winding with many overlapping turns, for example the TS. If the turns are close to each other, there is a finite capacitance between them. For high frequencies the leakage currents appear between particular turns and the current will be changed along the wire. This does not have any relation to the violation of the continuity equation div $\vec{j} = 0$, which will be fulfilled due to the presence of other $\vec{j}$ components having a direction different from that of wire. Since the current density changes along the wire, the time dependence is not now separated. This should lead to the violation of the reciprocity theorem. We conclude: the violation of the reciprocity is possible for high frequencies and a large number of overlapping coils.

In general, two windings with the same voltages at their terminals do not satisfy the reciprocity theorem if the time dependence is not separated in their charge-current densities and if these charge-current densities are different. The theoretical analysis of an experimental situation becomes easier if one of the windings is chosen as simple as possible (in particular, time dependence can be separated in its charge-current density), while the time dependence of other charge-current density should be non-separable. The measurement process involves two stages:

(1) Apply time-dependent voltage to the terminals of the first winding and measure an induced voltage at the terminals of the second winding;
(2) Apply the same time-dependent voltage to the terminals of the second winding and measure an induced voltage at the terminals of the first winding.

These induced voltages do not coincide if the time dependence in at least one of charge-current densities is not separable (despite the equality of applied voltages at the terminals of each winding).

At present we did not succeed in evaluating the explicit form of the resulting current with a non-separable space–time dependence (arising from the current leakages at high frequencies) for the realistic winding. Instead, in the next section, we consider the simplest charge–current density with non-separable space–time dependence and prove the violation of the Lorentz and Feld–Tai lemmas. We realize that this example is slightly unrealistic. It is needed to support the conclusion on the possible violation of reciprocity following from the alternative proof of the Lorentz and Feld–Tai lemmas given in section 7.3.

7.4.1. Concrete example of the reciprocity violation: interacting electric oscillator and current loop. We demonstrate the violation of the Lorentz and Feld–Tai lemmas using the interacting electric oscillator and an infinitesimal current loop as an example. We consider the electric oscillator oriented along the z axis:

$$\rho_{osc} = e\delta(x)\delta(y)\delta(z - a\cos\omega t)$$

$$j_z^{osc} = -ea\omega\sin\omega t\,\delta(x)\delta(y)\delta(z - a\cos\omega t). \tag{7.27}$$

Let the origin of a current loop br represented by the vector $\vec{r}_L$ and let $\vec{n}_L$ be a vector normal to its plane. Then, according to (3.16)

$$\vec{j}_L = f_L(t)\,\mathrm{curl}(\vec{n}_L\delta^3(\vec{r} - \vec{r}_L)). \tag{7.28}$$

We first evaluate the electromagnetic potentials of an electric oscillator. They are obtained from the general expressions

$$\Phi_{osc}(\vec{r}, t) = \int \frac{1}{R}\rho_{osc}(\vec{r}\,', t')\delta\left(t' - t + \frac{R}{c}\right)\mathrm{d}V'\,\mathrm{d}t'$$

$$A_z^{osc}(\vec{r}, t) = \int \frac{1}{R}j_z^{osc}(\vec{r}\,', t')\delta\left(t' - t + \frac{R}{c}\right)\mathrm{d}V'\,\mathrm{d}t'$$

$$R = |\vec{r} - \vec{r}\,'|$$

by substituting charge–current densities (7.27) into them and performing integration over space variables. This gives

$$\Phi_{osc}(\vec{r}, t) = e\int \frac{1}{R}\delta\left(t' - t + \frac{R}{c}\right)\mathrm{d}t'$$

$$A_z^{osc}(\vec{r}, t) = -e\beta_0\int \frac{\sin\omega t'}{R}\delta\left(t' - t + \frac{R}{c}\right)\mathrm{d}t'.$$

Here $\beta_0 = a\omega/c$, $R = [\rho^2 + (z - a\cos\omega t')^2]^{1/2}$ and $\rho^2 = x^2 + y^2$. Integrating over t' one obtains

$$\Phi_{osc}(\vec{r}, t) = \frac{e}{Q_1} \qquad A_z^{osc}(\vec{r}, t) = -\frac{e\beta_0\sin\omega t_1}{Q_1} \tag{7.29}$$

where $Q_1 = R_1 + \beta_0(z - a\cos\omega t_1)\sin\omega t_1$ and $R_1 = [x^2 + y^2 + (z - a\cos\omega t_1)^2]^{1/2}$. The retarded time t_1 is found from the equation

$$c(t - t_1) = R_1. \tag{7.30}$$

Since the charge velocity $\beta = -\beta_0\sin\omega t$ is less than one, there is only one root of (7.30).

When obtaining (7.29), the following property of the delta function was used: $\delta(f(x)) = \delta(x - x_1)/|f'(x_1)|$, where x_1 is the root of $f(x)$. For the treated case it looks like $\delta(t' - t + R/c) = \delta(t' - t_1)/[1 + \beta_0(z - a\cos\omega t_1)/R_1]$. The EMF strengths are given by

$$E_\rho = -\frac{\partial\Phi}{\partial\rho} \qquad E_z = -\frac{\partial\Phi}{\partial z} - \frac{\partial A_z}{c\partial t} \qquad H_\phi = -\frac{\partial A_z}{\partial\rho}.$$

When performing differentiation, one should take into account the fact that t_1 depends on the space–time coordinates of the observation point. The corresponding derivatives are given by

$$\frac{\mathrm{d}t_1}{\mathrm{d}t} = \frac{R_1}{Q_1} \qquad \frac{\mathrm{d}t_1}{\mathrm{d}\rho} = -\frac{\rho}{cQ_1} \qquad \frac{\mathrm{d}t_1}{\mathrm{d}z} = -\frac{z - a\cos\omega t_1}{Q_1}.$$

Then,

$$E_\rho^o = \frac{e\rho}{Q_1^3}\left[1 - \beta_0^2\sin^2\omega t_1 - \beta_0^2\cos\omega t_1\left(\frac{z}{a} - \cos\omega t_1\right)\right]$$

$$E_z^o = \frac{ea}{Q_1^3}\left[(1 - \beta_0^2\sin^2\omega t_1)\left(\frac{z}{a} - \cos\omega t_1 - \beta_0\frac{R_1}{a}\sin\omega t_1\right) + \beta_0^2\frac{\rho^2}{a^2}\cos\omega t_1\right]$$

$$H_\phi^o = -\frac{e\beta_0}{Q_1^3}\left[\beta_0\frac{R_1}{a}\cos\omega t_1 + (1 - \beta_0^2\sin^2\omega t_1)\sin\omega t_1\right]. \tag{7.31}$$

We need also the EMF strengths of the current loop. According to (3.17), they are given by

$$\vec{E}_L = \frac{1}{c^3r_L^2}\dot{D}_L((\vec{r} - \vec{r}_L)\times\vec{n}_L)$$

$$\vec{H}_L = \frac{1}{c^3r_L}\left[\frac{((\vec{r} - \vec{r}_L)\vec{n}_L)}{r_L^2}(\vec{r} - \vec{r}_L)F_L - \vec{n}_LG_L\right] \tag{7.32}$$

where $r_L = |\vec{r} - \vec{r}_L|$ and the argument of the D_L, F_L and G_L functions (see (3.18) for their definition) is $t - r_L/c$.

7.4.2. Lorentz lemma. Direct evaluation of the integrals entering into the Lorentz lemma gives

$$\mathcal{E}(osc, L) = \int \vec{j}_{osc}\vec{E}_L\,\mathrm{d}V = \frac{ea\omega\sin\omega t}{c^3R_L^2}\dot{D}_L[x_Ln_L^y - y_Ln_L^x]$$

where x_L, y_L and z_L define the position of the current loop; $R_L = [x_L^2 + y_L^2 + (z_L - a\cos\omega t)^2]^{1/2}$; the argument of the D_L function is $t - R_L/c$. Further,

$$\begin{aligned}\mathcal{E}(L, osc) &= \int \vec{j}_L\vec{E}_{osc}\,\mathrm{d}V\\ &= e\beta_0f_L(t)[x_Ln_L^y - y_Ln_L^x]\frac{R_{1L}}{cQ_{1L}}\frac{\mathrm{d}}{\mathrm{d}t_1}\\ &\quad\times\left\{\frac{1}{Q_{1L}^3}\left[\beta_0\frac{R_{1L}}{a}\cos\omega t_1 + (1 - \beta_0^2\sin^2\omega t_1)\sin\omega t_1\right]\right\}.\end{aligned}$$

Here $R_{1L} = [x_L^2 + y_L^2 + (z_L - a\cos\omega t_1)^2]^{1/2}$, $Q_{1L} = R_{1L} + \beta_0(z_L - a\cos\omega t_1)\sin\omega t_1$ and t_1 is found from the equation: $t - t_1 = R_{1L}/c$. Now we choose the time dependence of the charge–current loop to be the same as that of electric oscillator: $f_L(t) = f_0\sin\omega t$. Then, equalizing $\mathcal{E}(L, osc)$ and $\mathcal{E}(osc, L)$, one obtains

$$\begin{aligned}-\frac{k^2}{R_L^2}\left(\sin\omega t - \frac{1}{kR_L}\cos\omega t\right) &= \frac{R_{1L}}{cQ_{1L}}\frac{\mathrm{d}}{\mathrm{d}t_1}\\ &\quad\times\left\{\frac{1}{Q_{1L}^3}\left[\beta_0\frac{R_{1L}}{a}\cos\omega t_1 + (1 - \beta_0^2\sin^2\omega t_1)\sin\omega t_1\right]\right\}.\end{aligned}$$

Here $k = \omega/c$. This equality is not satisfied, and therefore the Lorentz lemma is not fulfilled for the interacting electric oscillator and current loop. Experimentally, this means that the non-coincidence of the voltages induced and the absence of the receiver–transmitter symmetry (see section 7.3.3). Instead, the more complicated relations (7.24) and (7.25) should be fulfilled. Their physical meaning is not so clear as that of the Lorentz lemma.

7.4.3. Feld–Tai lemma. For the integrals entering into Feld–Tai lemma one gets

$$\mathcal{H}(osc, L) = -\frac{ea\omega \sin\omega t}{c^3 R_L}\Big\{[(a\cos\omega t - z_L)n_L^z - x_L n_L^x - y_L n_L^y]\frac{a\cos\omega t - z_L}{R_L^2}F_L - n_L^z G_L\Big\}$$

$$\mathcal{H}(L, osc) = \frac{ef_L(t)}{c}\frac{R_{1L}}{Q_{1L}}\frac{\mathrm{d}}{\mathrm{d}t_1}\Big\{(x_L n_L^x + y_L n_L^y) \times \Big[1 - \beta_0^2\sin^2\omega t_1 - \beta_0^2\cos\omega t_1\left(\frac{z_0}{a} - \cos\omega t_1\right)\Big] + a n_L^z\Big[(1-\beta_0^2\sin^2\omega t_1)\left(\frac{z_l}{a} - \cos\omega t_1 - \beta_0\frac{R_{1L}}{a}\sin\omega t_1\right) + \beta_0^2\frac{\rho_L^2}{a^2}\cos\omega t_1\Big]\Big\}$$

$$\rho_L^2 = x_L^2 + y_L^2.$$

Putting $f_L(t) = f_0 \sin\omega t$ we find that the Feld–Tai lemma cannot be satisfied for the interacting electric oscillator and current loop.

It should be mentioned that the more general reciprocity relations formulated in section 7.3.4 are fulfilled for arbitrary time dependences and, in particular, for the interacting electric oscillator and current loop.

7.5. Historical remarks to section 7

Conditions (7.5) and (7.8), ensuring the validity of action and reaction and the fulfillment of the Lorentz and Feld–Tai lemmas for arbitrary interacting electromagnetic sources, are new. The same is valid for the interaction laws (7.12), (7.14) and (7.16) between the even and odd toroidal sources and for the generalizations (7.25) and (7.26) of the Lorentz and Feld–Tai lemmas. The conditions under which the Lorentz and Feld–Tai lemmas can be violated and the concrete example demonstrating this fact have never before been obtained.

8. Discussion and Conclusion

Recently, we were aware of experiments with toroidal coils of higher orders (see section 5). In these experiments, the non-coincidence of the voltages induced in the toroidal coils (i.e. the violation of the transmitter–receiver symmetry mentioned in section 7.3.3) was observed for large frequencies. The reciprocity theorem seems to be so well established that the scientists performing these experiments did not dare to attribute this non-coincidence to its violation. However, the present consideration shows that this violation is, indeed, possible. It should be mentioned that the violation of reciprocity will lead to serious consequences in both theoretical and experimental electromagnetism. According to [13]:

> One of the basic and most important theorems of electromagnetic theory is the so-called reciprocity theorem. Its importance is evident from its wide range of applicability in all branches of electrical engineering.

We briefly enumerate the main results obtained.

(1) We obtained expressions describing the EMFs of a current loop, TS and electric dipole with a periodic current in their windings. These expressions are valid for arbitrary distances and frequencies. We did not find them in the available textbooks and journal papers (only long-wave limit expressions were found in the literature). Various particular cases are considered and conditions for their validity are given. The interaction of these sources with external EMF and between themselves is found.

(2) We applied the reciprocity theorem (Lorentz and Feld–Tai lemmas) to the EMFs of time-dependent electric dipole, current loop, TS and higher-order EMF sources. It is shown that the proportionality of time derivatives of the EMF strengths to the EMF strengths themselves is not a necessary condition for the fulfillment of the reciprocity theorem.

(3) An alternative proof of the reciprocity theorem is given. It is shown that the reciprocity theorem works for more general time dependences than previously suggested. The conditions for its validity are reduced to the following two:

(i) the time dependence should be separated from the spatial dependence in the charge–current densities of interacting sources;
(ii) the time dependences of these sources should be the same.

These conditions are essentially the same as those needed for the equality of action and reaction between two interacting electromagnetic sources. The estimation of action–reaction violation for an interacting current loop and TS is given.

(4) Conditions under which the reciprocity theorem can be violated are given. A concrete example is presented for which the reciprocity theorem is manifestly violated.

(5) New reciprocity-like theorems valid for arbitrary space–time dependences (and, in particular, for those discussed in the previous item) of the interacting current densities are obtained. However, their physical meaning is not very clear.

Acknowledgments

The author is deeply indebted to Professor N I Zheludev from Southampton University, England, and to Richard Wardle from the Southampton Chilworth Research Centre who, in fact, initiated this investigation.

Appendix

We begin with the well known relation (see, e.g., [20])

$$\cos\nu\theta J_\nu(k\sqrt{d^2 + R^2 - 2dR\cos\psi}) = \sum_{-\infty}^{\infty} J_m(kR)J_{m+\nu}(kd)\cos m\psi \qquad R < d \tag{A.1}$$

where $\tan\theta = R\sin\psi/(d - R\cos\psi)$. For $R \ll d$, the angle θ may be put to zero. Then,

$$J_\nu(k\sqrt{d^2 + R^2 - 2dR\cos\psi}) \approx \sum_{-\infty}^{\infty} J_m(kR)J_{m+\nu}(kd)\cos m\psi \qquad R \ll d. \tag{A.2}$$

We cannot put $R = 0$ in the RHS of this equation, since for high frequencies kR may be large. Furthermore,

$$j_{2n+1}(ky) = \sqrt{\frac{\pi}{2ky}}J_{2n+3/2}(ky) \approx \sqrt{\frac{\pi}{2kd}}J_{2n+3/2}(ky)$$

where $y = (d^2 + R^2 + 2dR\cos\psi)^{1/2}$ is the same as in (3.22). We changed y by d outside the Bessel function. This is possible since $R \ll d$. Then, according to (A.1),

$$j_{2n+1}(ky) \approx \sum_{-\infty}^{\infty}(-1)^m J_m(kR) J_{2n+1+m}(kd)\cos m\psi \qquad \text{(A.3)}$$

$$R \ll d.$$

Therefore, the integral defining D_{2n+1} is given by

$$\int_0^{2\pi} j_{2n+1}(ky)\sin^2\psi\,\mathrm{d}\psi$$
$$= \pi\left\{J_0(kR)j_{2n+1}(kd) - \frac{1}{2}J_2(kR)[j_{2n+3}(kd) + j_{2n-1}(kd)]\right\}. \qquad \text{(A.4)}$$

This exactly coincides with (3.29).

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Arnout Ceulemans

Topological Invariants in Molecular Networks and their Current-like Observables

Super-Toroidal Electrodynamics Workshop, Southampton
November 5, 2004

Outline

Introduction: topological invariants and molecular networks

Case study: the rotating electron in a circular magnetic field

Case study: anapole moments in toroidal molecular networks

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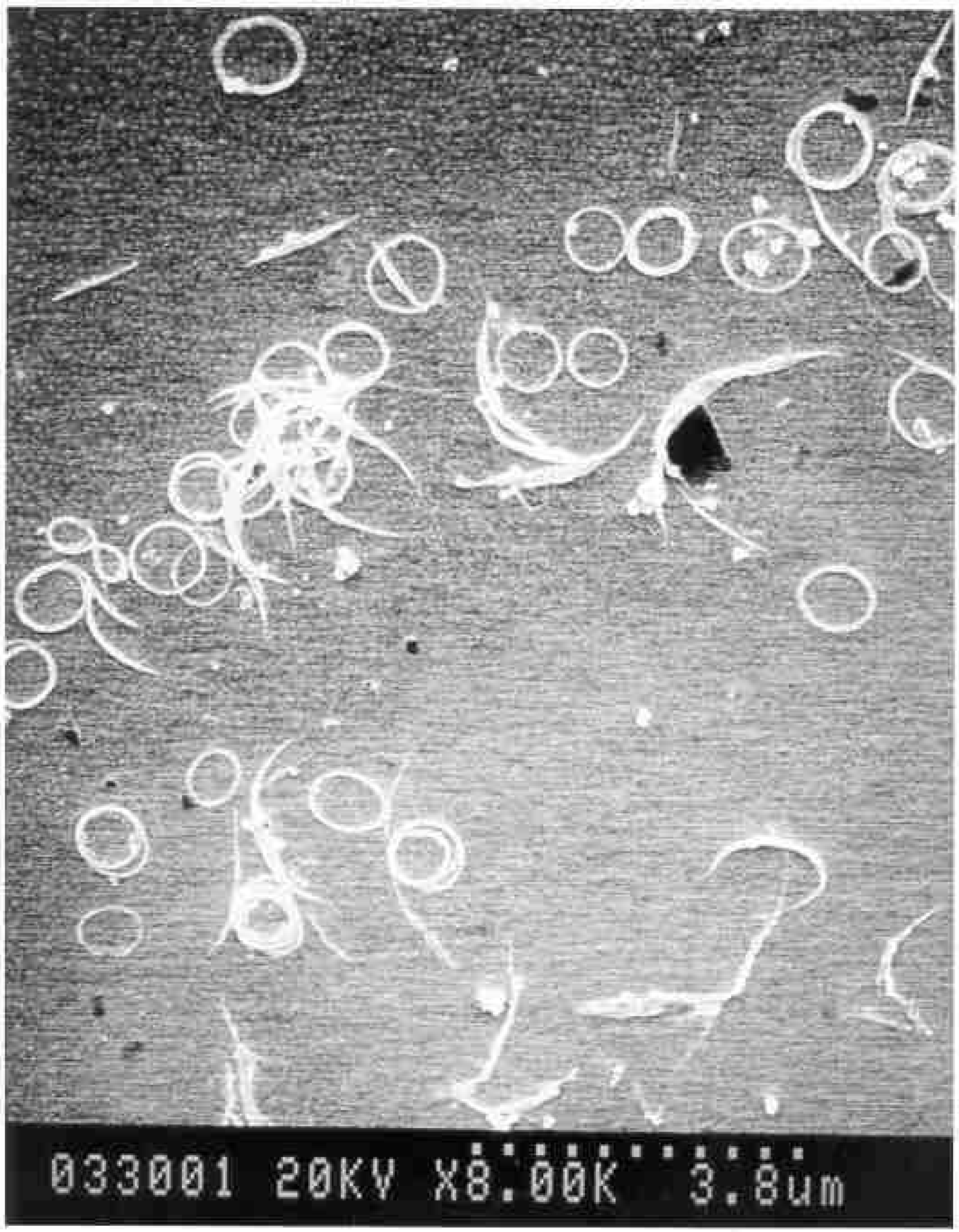


SEM image of multi-walled nanotube coils on a silicon substrate

Ring diameter: 700 nm
Tube diameter: 1.4 nm

(Avouris, IBM)

AFM micrograph
of 1 μm ring placed
over gold electrodes

(Avouris, IBM)

Euler theorem (± 1750)

- The number of vertices (*v*) minus edges (*e*) plus faces (*f*) in a polyhedron is a constant that is characteristic for the surface on which the polyhedron can be embedded. For a polyhedron that can be mapped on a sphere (S_0) one has:

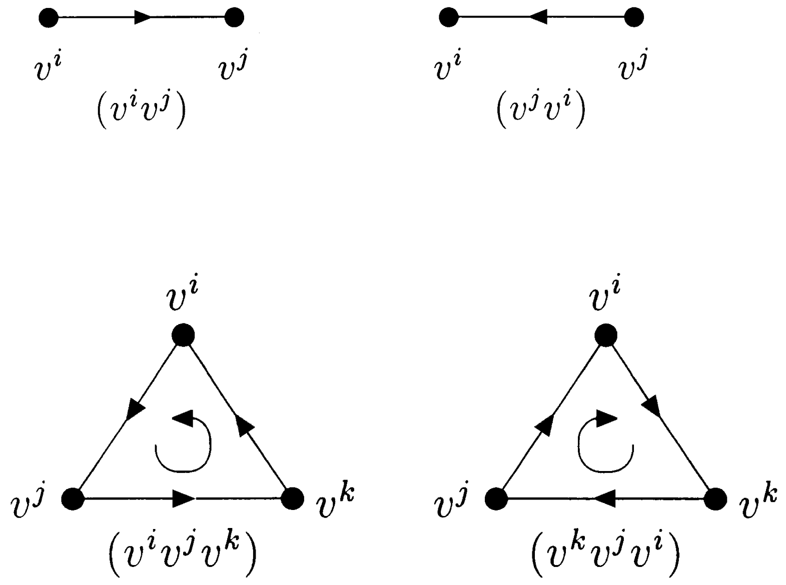
$$v - e + f = \chi(S_0) = 2$$

From geometry to topology: the polyhedral complex

Vertices: dimensionality: zero
functionality: scalar

Edges: dimensionality: one
functionality: vector

Faces: dimensionality: two
functionality: rotor



Symmetry extension of Euler's theorem:

$\Gamma_{\sigma}(v)$: induced symmetry representation of vertex points

$\Gamma_{\uparrow}(e)$: induced symmetry representation of edge vectors

$\Gamma_{\forall}(f)$: induced symmetry representation of face rotors

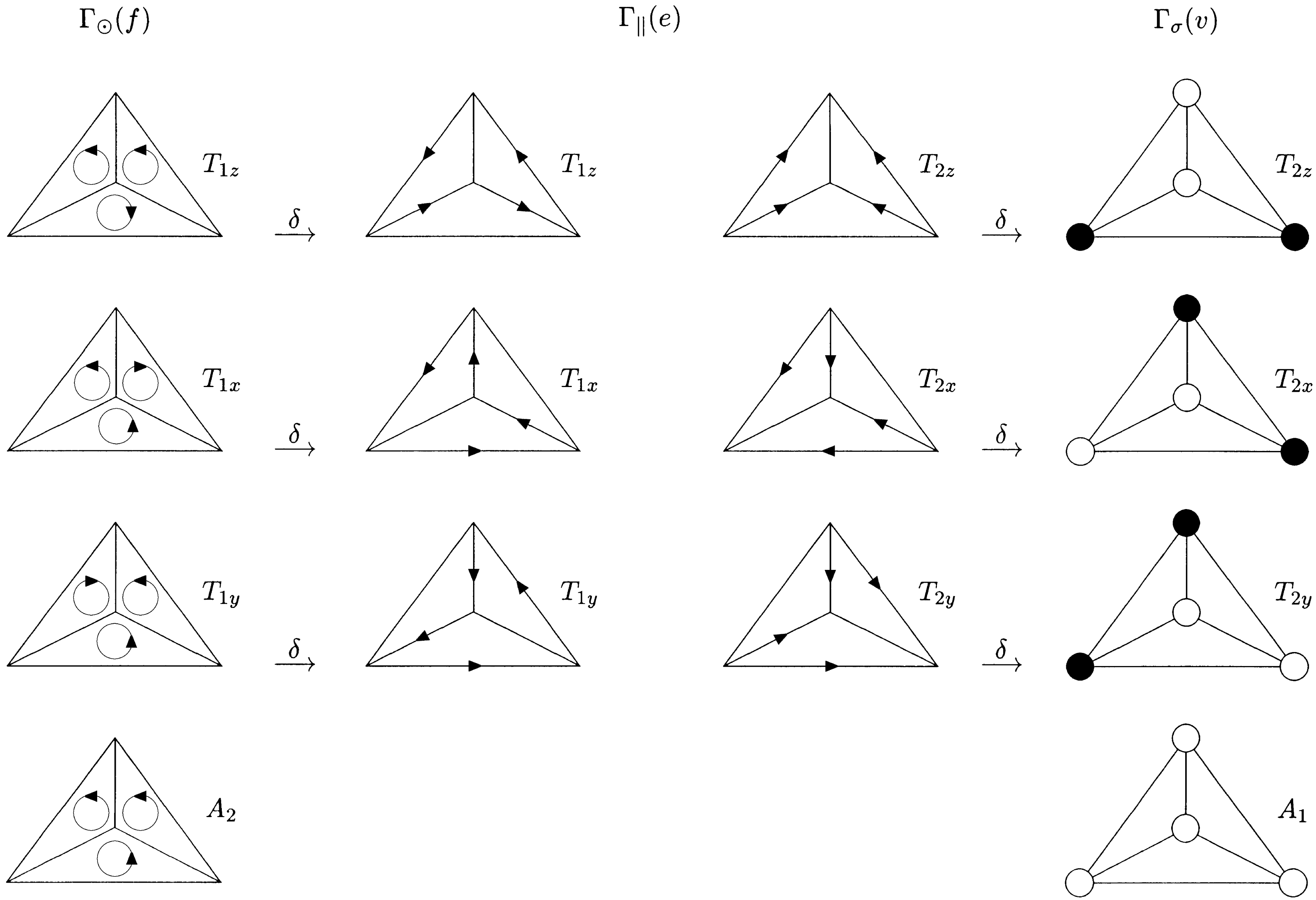
Γ_{0} : totally symmetric representation

Γ_{ε} : pseudoscalar representation

$$S_0: \ \Gamma_{\sigma}(v) - \Gamma_{\uparrow}(e) + \Gamma_{\forall}(f) = \Gamma_{0} + \Gamma_{\varepsilon}$$

A. Ceulemans, P.W. Fowler, *Nature*, 1991

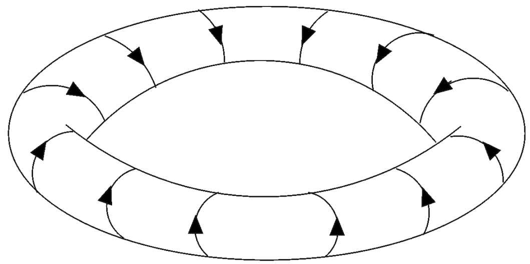
Example: tetrahedron



Embedding on a torus: $\chi(S_1) = 0$

$$S_1:\ \Gamma_\sigma(v) - \Gamma_{\uparrow}(e) + \Gamma_\forall(f) = \Gamma_0 - \Gamma_{T_z} - \Gamma_{R_z} + \Gamma_\varepsilon$$

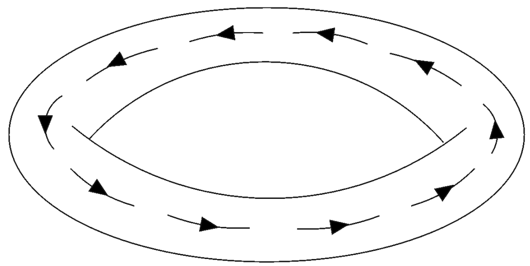
$\Gamma_0(\Sigma_g^+)$: Electric monopole



$\Gamma(T_Z)$

$\Gamma_{T_z}(\Sigma_u^+)$: Electric dipole, *anapole*

$\Gamma_{R_z}(\Sigma_g^-)$: Magnetic dipole



$\Gamma(R_Z)$

$\Gamma_\varepsilon(\Sigma_u^-)$: Magnetic monopole

Anapole moment operator

$$a_z = -\frac{1}{2}\left(M_{xy} - M_{yx}\right)$$

with:

$$\mathrm{M}_{\alpha\beta} = -2\frac{dV}{d\left(\partial\nabla_{\beta}B_{\alpha}\right)_0}$$

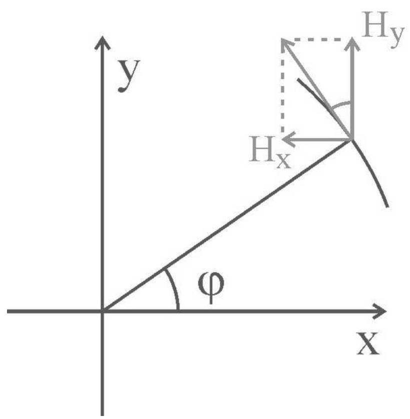
The anapole moment operator is the antisymmetric combination of the second-order magnetic moments operators. This describes the interaction of the system with the rotor of the external field.

For B’ be a uniform rotor field along z-direction:

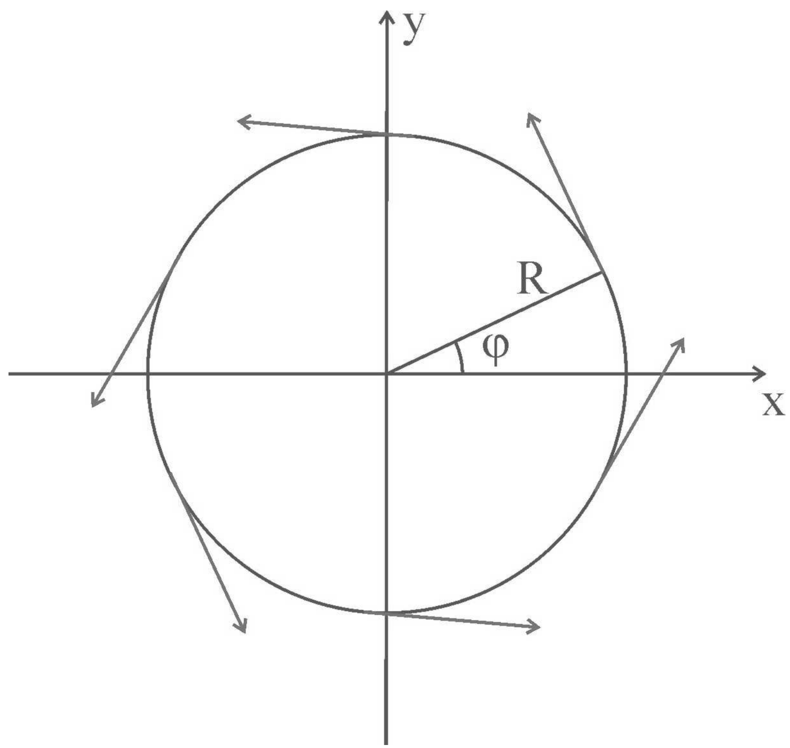
- Anapole moment: $$a_z = -2\frac{\delta V}{\delta B'}$$

- Anapole susceptibility: $$A = -2\frac{\partial^2 V}{(\partial B')^2}$$

Case 1: particle on a ring in a circular magnetic field



$$H_x = -H \sin\varphi$$
$$H_y = H \cos\varphi$$



Hamiltonian

$$H = -\frac{h^2}{8\pi^2 I}\frac{\partial^2}{\partial\varphi^2} - \mu_B H\{-\sin\varphi\ \hat{s}_x + \cos\varphi\ \hat{s}_y\}$$

Rotational symmetry

$$[H, \frac{h}{2\pi i}\frac{\delta}{\delta\varphi} + \hat{s}_z] = 0$$

Ansatz

$$\Psi_l = c_{1l}\ e^{il\varphi}\,|\,\alpha > + c_{2l}\ e^{i(l+1)\varphi}\,|\,\beta >$$

$$(l_z + s_z)\Psi_l = (l + \frac{1}{2})\Psi_l$$

Solutions:

Energy matrix:

$$\begin{pmatrix} \frac{\hbar^2}{2I} l^2 - E & \frac{i}{2}\hbar\mu_B H \\ -\frac{i}{2}\hbar\mu_B H & \frac{\hbar^2}{2I}(l+1)^2 - E \end{pmatrix}$$

Eigenvalues

$$\mathrm{E}_{\pm}(l) = \frac{\hbar^2}{2I}(l^2 + l + \frac{1}{2}) \ \pm \ \frac{1}{2}\sqrt{\frac{\hbar^4}{I^2}(l + \frac{1}{2})^2 + \hbar^2 \mu_B^2 H^2}$$

Second-order magnetic moment

$$m_{xy} = \sum_i \frac{e}{2m}\left[r_{iy}\left(\frac{2}{3}\hat{l}_{ix} + g\hat{s}_{ix}\right) + \left(\frac{2}{3}\hat{l}_{ix} + g\hat{s}_{ix}\right) r_{iy} \right]$$

Magnetic anapole moment

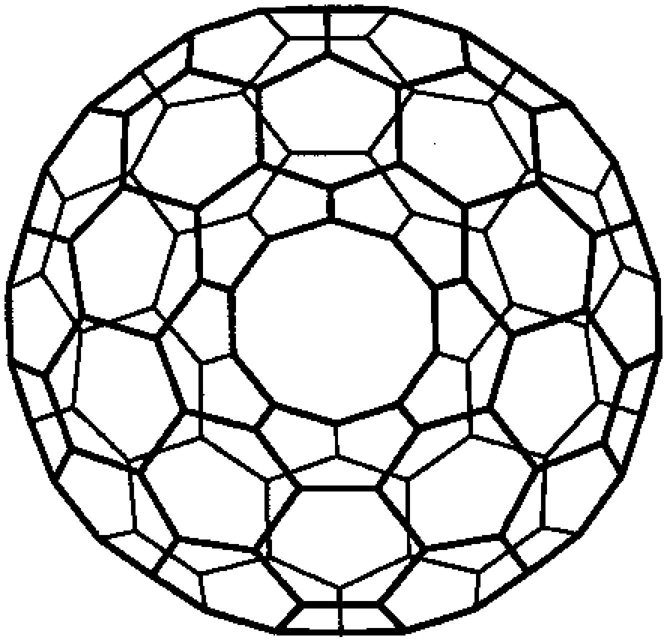
$$\hat{a}_z = -\frac{1}{2}\left[m_{xy} - m_{yx} \right] = -\sum_i \frac{e}{2m}\left(\hat{s}_i \times r_i \right)_z$$

Expectation value

$$\left\langle \hat{a}_z \right\rangle_{\pm l} \approx \mp \frac{eR}{4m} \frac{\mu_B I H}{\hbar\left(l + 1/2\right)}$$

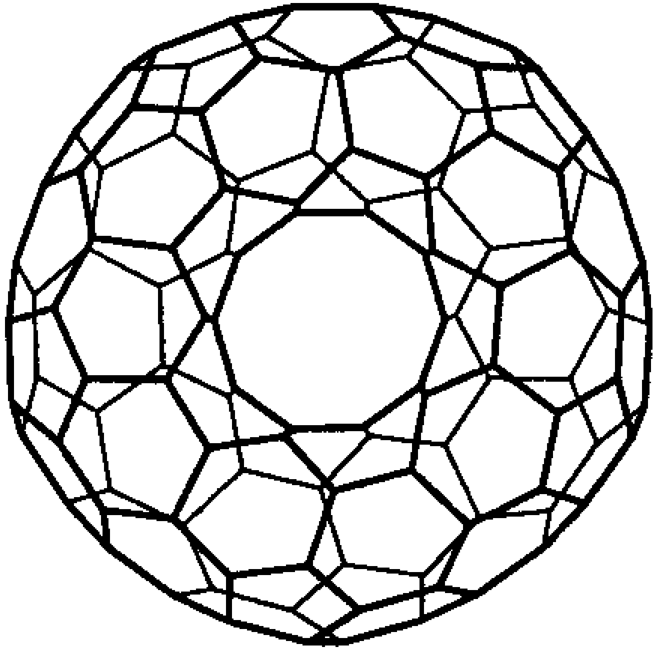
Case 2: Molecular network

Model: 120 atom carbon toroid



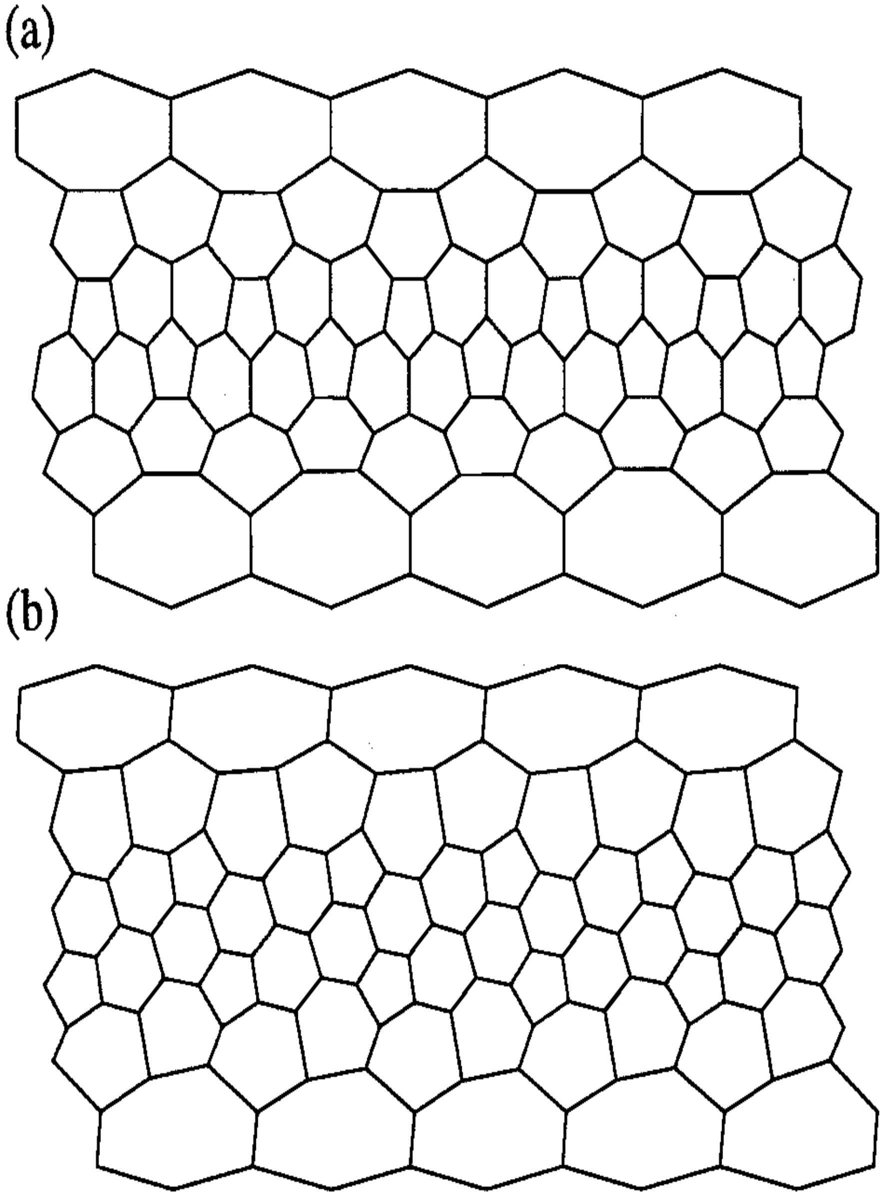
(a)

Achiral: D_{5d}



(b)

Chiral: D_5



Projections of the two networks, obtained by unpeeling the toroidal surface. Top and bottom edges represent the inner equatorial circuit of the torus.

Hückel treatment and London approximation

$$h_{kl} = \beta_{kl} \exp(2\pi i f_{kl}) \quad \text{with: } f_{kl} = \frac{e}{2hc}\left(\vec{A}_k - \vec{A}_l\right)\left(\vec{R}_k + \vec{R}_l\right)$$

Vector Potentials:

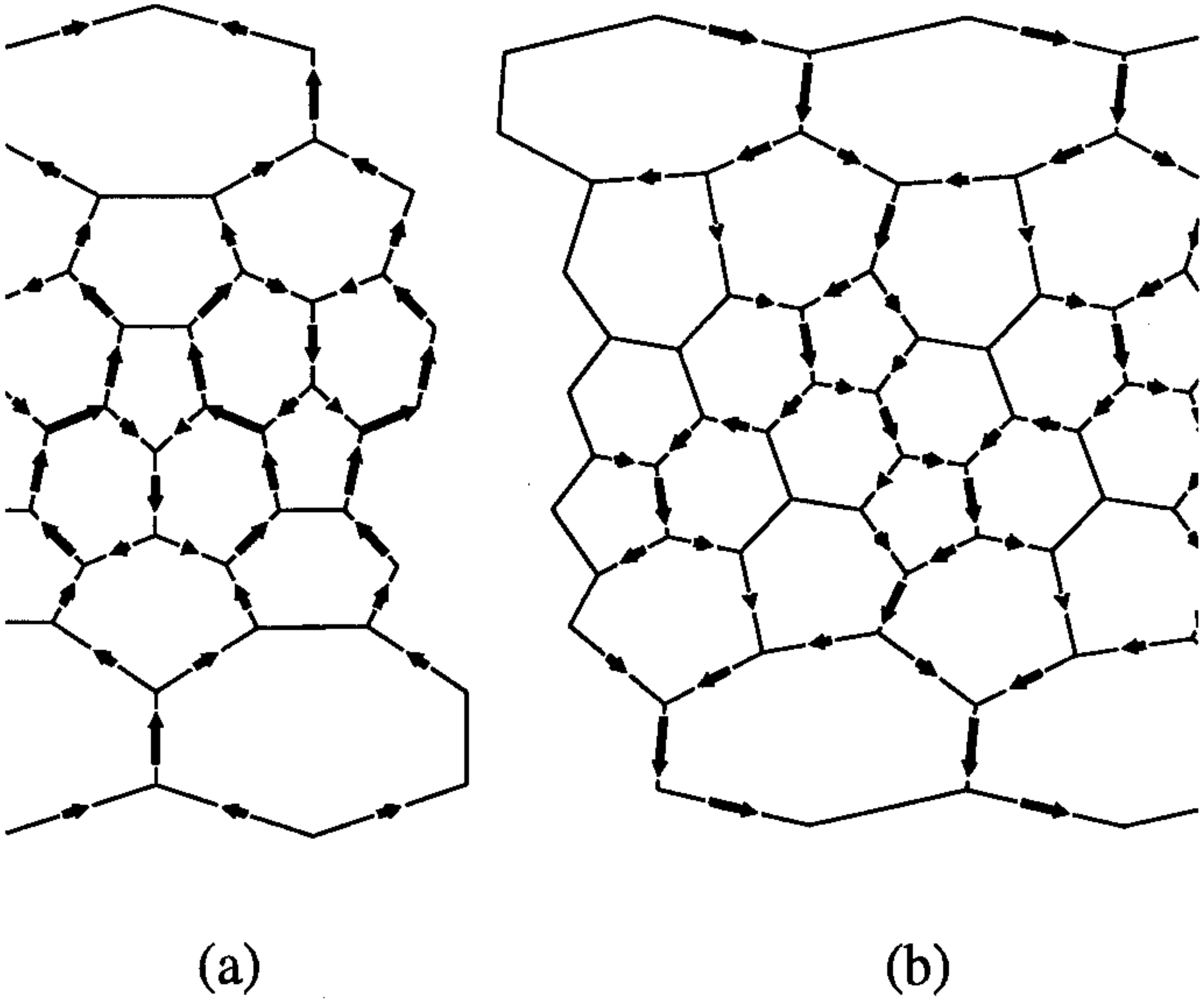
- Uniform vector field: $\vec{A} = \frac{1}{2}\ \vec{r} \times \vec{B}$

- Uniform rotor field: $\vec{A}' = \frac{1}{6}\ \vec{r} \times \vec{r} \times \vec{B}'$

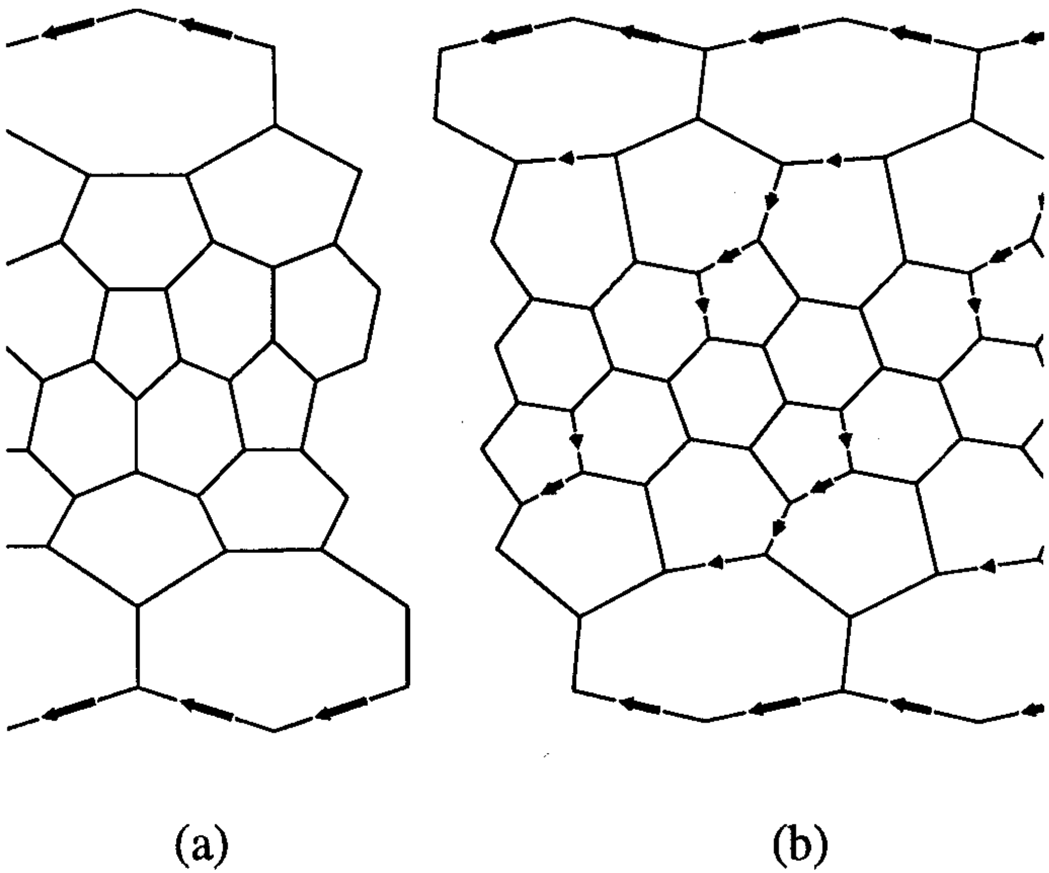
Calculated dipole and anapole properties

For details: see A.Ceulemans et al, PRL, 80, 1861 (1998)

	D_{5d}	D_{5d}	D_5	D_5
	neutral	cation	neutral	cation
χ	-2200	-1890	-1130	-1180
A	5420	8000	-12600	-9350
m_z	0	2.74	0	4.86
a_z	0	0	0	13.9
M	0	0	-265	-380



Induced currents under a uniform magetic field along the central axis



Current states for open shell toroidal cations. The dominant circulation is around the inner equator of the torus. Note the mixed dipolar-anapolar character of the flow in the chiral torus.

Conclusions

- The toroidal topology has two current-like invariants with the symmetries of magnetic and electric dipole
- A spin in a uniform rotor field has a toroidal moment
- Chiral toroidal networks can couple a magnetic dipole and anapole

Toroidal moments and atomic emission in condensed media

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Russia

Toroidal moments

- Static toroidal moments

Static toroidal moments cause parity nonconservation in atoms arising from the *P*-odd part of the weak interactions.

C.S.Wood *et al.* (Science **275** (1977) 1759) measured a static dipole toroidal moment of the ^{133}Cs nucleus in the atomic $6S \to 7S$ transition in 1997.

- Toroidal moments of transitions

Toroidal moments of transitions could exist even if a system does not have a static toroidal moments.

For a very limited number of nuclei, the toroidal moments of nuclear E1 transitions can be extracted from data on anomalous internal electronic conversion of γ-rays - M.A.Listengarten *et al*., Izv. Akad. Nauk SSSR, Ser. Fiz. **45** (1981) 2038.

How to extract the toroidal moments of transitions from the half-lifes of atomic levels?
There are two main ideas

- Decay probability of excited atomic states depends on electronic properties of the bulk ε and μ as

$$W_{medium}^{EL} = f_L^2(\varepsilon)\varepsilon^{L-1/2}\mu^{L+1/2}W_{vac}^{EL}$$

- Formfactors (moments) of the electrical transitions contain toroidal formfactors (moments) with a factor k^2

  ($k = (\varepsilon\mu)^{1/2}\omega$) :

$$< J_f M_f \,|\, \hat{Q}_{LM}^{E}(k) \,|\, J_i M_i > = i\omega < J_f M_f \,|\, \hat{Q}_{LM}^{C} \,|\, J_i M_i >$$

$$+ ik^2 < J_f M_f \,|\, \hat{Q}_{LM}^{T}(k) \,|\, J_i M_i >$$

Electromagnetic Decay in a Medium

An infinite dielectric medium influences the probability of spontaneous emission in the optical region. The probability of electric dipole transition in a medium with dielectric constant ε at an emission frequency ω can be expressed through the probability of spontaneous decay in a vacuum by the relationship from the vacuum values according to the formula

$$W_{medium}^{E1} = f^{2}(\varepsilon)\varepsilon^{1/2} W_{vac}^{E1} \qquad (1)$$

The function $f(\varepsilon)$ relates the electric component $\mathbf{E}_m$ of a macroscopic electromagnetic field in a medium to the local electric field $\mathbf{E}_{loc}$ at the point where dipole is located

$$\mathbf{E}_{loc} = f(\varepsilon)\,\mathbf{E}_m$$

How the factor $\varepsilon^{1/2}$ arises in Eq. (1)?

Electromagnetic Decay in a Medium

Expression for the probability of electric dipole transition:

$$W^{E1}_{medium} = 2\pi \left| \left\langle f \left| \hat{\vec{d}} \cdot \hat{\vec{E}}^{+}_{loc} \right| i \right\rangle \right|^2 \rho_{medium}(\omega)$$

where $\hat{\vec{d}}$ is the dipole moment operator of the emitting system

and $\hat{\vec{E}}^{+}_{loc}$ is the electric-field creation operator.

The field operator $\hat{\vec{E}}^{+}_{m}$ and the density $\rho_{medium}(\omega)$ of photon final state are renormalized from the vacuum values according to the formulas

$$\hat{\vec{E}}^{+}_{m} = \frac{1}{\varepsilon^{1/2}} \hat{\vec{E}}^{+}_{vac} \quad , \qquad \rho_m(\omega) = \varepsilon^{3/2} \rho_{vac}(\omega)$$

Electromagnetic Decay in a Medium

The first relation follows from the quantization rules for the electromagnetic field in a medium. For the gauge divA = 0, the equation

$$\Delta\vec{A} - \varepsilon\partial_t^2\vec{A} = 0$$

for the vector potential **A** follows from the Maxwell equations in a uniform dielectric medium with permeability μ = 1 in the absence of extrinsic currents and charges:

$$\mathrm{curl}\mathbf{E} = -\star_t\mathbf{H} \qquad \mathrm{curl}\mathbf{H} = \star_t\mathbf{D} \qquad \mathbf{D} = \varepsilon\mathbf{E}$$

$$\mathrm{div}\mathbf{H} = 0 \qquad \mathrm{div}\mathbf{D} = 0$$

where the electric and magnetic fields are defined via **A** in the standard way:

$$\mathbf{E} = -\star_t\mathbf{A} \qquad \mathbf{H} = \mathrm{curl}\mathbf{A}$$

Quantization of the Electromagnetic Field in a Medium

The vector potential can be written as an expansion in plane waves

$$\hat{\vec{A}}(\vec{r},t)=\sum_{\vec{k}}\sum_{\lambda=1,2}\left(\hat{a}_{\vec{k},\lambda}\vec{A}_{\vec{k},\lambda}e^{-i\omega t}+\hat{a}^{+}_{\vec{k},\lambda}\vec{A}^{*}_{\vec{k},\lambda}e^{i\omega t}\right)$$

where

$$\vec{A}_{\vec{k},\lambda}=\vec{e}_{\vec{k},\lambda}\sqrt{\frac{2\pi}{\varepsilon\omega}}e^{i\vec{k}\vec{r}}$$

$\boldsymbol{e}_{\boldsymbol{k}\lambda}$ is the unit vector of plane wave polarization. The factor $(2\pi/\varepsilon\omega)^{1/2}$ follows from the formula for the energy of a free electromagnetic field in a medium $1/8\pi\int(\mathbf{E}_m\mathbf{D}_m+\mathbf{H}_m\mathbf{B}_m)d^3r$

The operators of photon creation and annihilation in obey the ordinary commutation relations

$$[\hat{a}_{\vec{k},\lambda}\hat{a}_{\vec{k}',\lambda'}]=[\hat{a}^{+}_{\vec{k},\lambda}\hat{a}^{+}_{\vec{k}',\lambda'}]=0 \qquad [\hat{a}_{\vec{k},\lambda}\hat{a}^{+}_{\vec{k}',\lambda'}]=\delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'}$$

and the field energy and momentum operators are expressed in terms of the creation and annihilation operators in the standard form

$$\hat{H}=\sum_{\vec{k}}\sum_{\lambda=1,2}\omega\left(\hat{a}^{+}_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}+\frac{1}{2}\right) \qquad \hat{\vec{P}}=\sum_{\vec{k}}\sum_{\lambda=1,2}\vec{k}\hat{a}^{+}_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}$$

Electromagnetic Decay in a Medium

The explicit form of the creation operator for an electric field with momentum ***k*** and energy ω in a medium is

$$\hat{\vec{E}}^{+}_{\vec{k},\lambda\,medium} = -i\vec{e}^{\,*}_{\vec{k},\lambda}\sqrt{\frac{2\pi\omega}{\varepsilon}}e^{-i\vec{k}\vec{r}}\,\hat{a}^{+}_{\vec{k},\lambda}$$

The renormalization of phase volume: in matter

$$\boldsymbol{k}^2 = \varepsilon\omega^2 ,$$

and the $k^2dk/d\omega$ value increases in a medium by a factor of $\varepsilon^{3/2}$.

The result: $\varepsilon^{1/2}$ -type dependence of the $E1$ emission probability on the dielectric constant.

Toroidal moments of transition

The interaction Hamiltonian for the emission has the form

$$H_{\text{int}}(t) = -e\int j_{fi}^{\nu}(\vec{r},t)A_{\nu}(\vec{r},t)d^3r$$

where *e* is the electron charge. The current $\boldsymbol{j}_{\text{fi}}$ and the vector-potential of the radiation field A_ν are in the interaction picture.

The Hamiltonian for the interaction in the *EL* mode is proportional to

$$\int d^3r\vec{A}_{LM}^{E^*}(k,\vec{r})\cdot\vec{j}_{fi}(\vec{r})$$

Toroidal moments of transition

The explicit form for the electric multipole field is

$$\vec{A}_{LM}^{E^*}(k,\vec{r}) = \sqrt{\frac{L+1}{2L+1}}\, j_{L-1}(kr)\vec{\mathrm{Y}}_{LM}^{L-1}(\vec{r}) - \sqrt{\frac{L}{2L+1}}\, j_{L+1}(kr)\vec{\mathrm{Y}}_{LM}^{L+1}(\vec{r})$$

where $j_L(kr)$ are spherical Bessel functions, and $\boldsymbol{Y}_{Jm}{}^{L}(\boldsymbol{r})$ are spherical vector functions.

Usually one introduces the electric multipole form-factors

$$< J_f M_f \,|\, \hat{Q}_{LM}^{E} \,|\, J_i M_i >= \frac{(2L+1)!!}{k^L}\left(\frac{L}{L+1}\right)^{1/2}\left(\frac{4\pi}{2L+1}\right)^{1/2}\int d^3 r \vec{A}_{LM}^{E^*}(k,\vec{r})\cdot \vec{j}_{fi}(\vec{r})$$

Toroidal moments of transition

and uses long-wave approximation (*kr* << 1) for derivation of the relation

$$\lim_{k\to 0} < J_f M_f \,|\, \hat{Q}^E_{LM}(k) \,|\, J_i M_i > = i\omega_{fi} < J_f M_f \,|\, \hat{Q}^C_{LM} \,|\, J_i M_i >$$

where the standard formula is used for the charge multipole moments

$$< J_f M_f \,|\, \hat{Q}^C_{LM} \,|\, J_i M_i > = \left(\frac{4\pi}{2L+1}\right)^{1/2} \int d^3 r \rho_{fi}(\vec{r}) r^L Y^*_{LM}(\vec{r})$$

Such an approximation is not correct, because the toroidal moments are lost (V.M.Dubivik and A.A.Cheshkov, Sov. J. Part. Nucl. **5** (1974) 318). One obtains a stricter representation for the electric multipole form factor if takes into account higher-order terms in the Bessel function's expansion into a power series.

Toroidal moments of transition

Toroidal multipole form factors are introduced according to the relation

$$< J_f M_f \,|\, \hat{Q}^E_{LM}(k) \,|\, J_i M_i >= i\omega_{fi} < J_f M_f \,|\, \hat{Q}^C_{LM} \,|\, J_i M_i >$$

$$+ik^2 < J_f M_f \,|\, \hat{Q}^T_{LM}(k) \,|\, J_i M_i >$$

The explicit form for toroidal moments is easily obtained.

$$< J_f M_f \,|\, \hat{Q}^T_{LM} \,|\, J_i M_i >= \frac{i}{2}\left(\frac{4\pi}{2L+1}\right)^{1/2}\left(\frac{L}{L+1}\right)^{1/2}$$

$$\times\int d^3r \vec{j}_{fi}(\vec{r}) r^{L+1}\left(\vec{\mathrm{Y}}^{L-1\,*}_{LM}(\vec{r}) + \frac{2}{2L+3}\sqrt{\frac{L}{L+1}}\vec{\mathrm{Y}}^{L+1\,*}_{LM}(\vec{r})\right)$$

Toroidal moments of transition

This result enables one to make a more accurate parametrization in the Hamiltonian for radiation of the *EL* mode in the range $kr << 1$:

$$\int d^3 r \vec{A}^{E^*}_{LM}(k,\vec{r}) \cdot \vec{j}_{fi}(\vec{r}) = \frac{ik^L}{(2L+1)!!}\left(\frac{L+1}{L}\right)^{1/2}\left(\frac{2L+1}{4\pi}\right)^{1/2}$$

$$\times\left(\frac{\omega}{k} < J_f M_f \mid \hat{Q}^C_{LM} \mid J_i M_i > + k < J_f M_f \mid \hat{Q}^T_{LM} \mid J_i M_i >\right)$$

Electric multipole emission is possible even if $\rho_{fi} = 0$, i.e., if all charge multipole moments are equal to zero.

Toroidal moments of transition

The final formula for the *EL* emission in a nonabsorbing medium is

$$W_{medium}^{EL} = f_L^2(\varepsilon)\varepsilon^{L-1/2}\mu^{L+1/2}W_{vac}^{EL(C)} \times (1+$$

$$+2\omega\varepsilon\mu \frac{\mathrm{Re} < J_f \| \hat{Q}_L^T \| J_i > \mathrm{Re} < J_f \| \hat{Q}_L^C \| J_i > + \mathrm{Im} < J_f \| \hat{Q}_L^T \| J_i > \mathrm{Im} < J_f \| \hat{Q}_L^C \| J_i >}{|< J_f \| \hat{Q}_L^C \| J_i >|^2}$$

$$+\omega^2\varepsilon^2\mu^2 \frac{|< J_f \| \hat{Q}_L^T \| J_i >|^2}{|< J_f \| \hat{Q}_L^C \| J_i >|^2})$$

Ratio between toroidal and charge terms in the emission probability

The toroidal term becomes equal approximately to the charge term in the range ♏µ~1/ω*a*, where the *a* is a characteristic size of the emitting system. If an atom emits an optical range photon, then $a \sim a_B$, where a_B is the Bohr radius. Inside a dielectric medium, for example, in the real cavity model the known linear refractive-index dependence

$$W_m^{E1}/W_{\mathrm{vac}}^{E1} \sim n$$

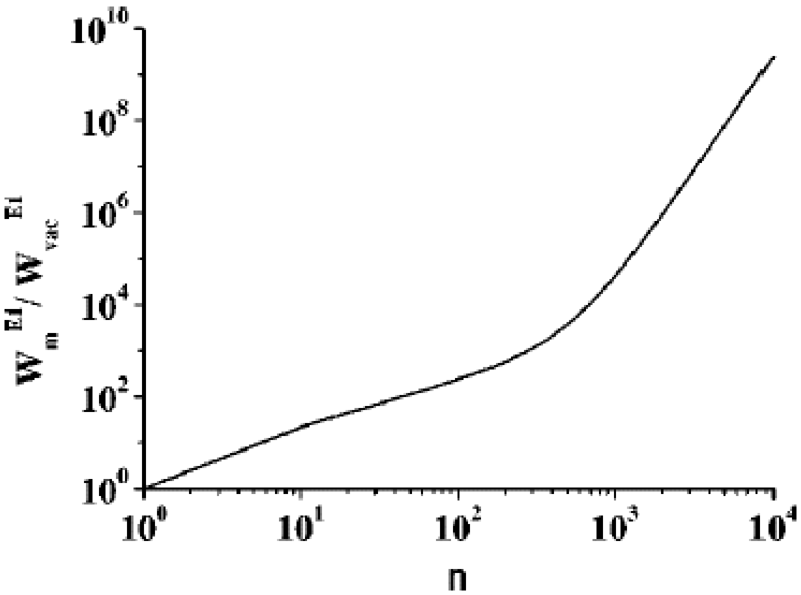
becomes

$$W_m^{E1}/W_{\mathrm{vac}}^{E1} \sim n^5$$

in the range $n \geq 1/(\omega\, a_B)^{1/2}$.

$W_m^{E1}/W_{\text{vac}}^{E1}$ as a function n

FIG. 1. Plot of $W_m^{E1}/W_{\text{vac}}^{E1}$ as a function of the refractive index n for a typical atomic transition in the optical range (ω = 2-3 eV, or λ = 400-600 nm). The plot corresponds to a real cavity.



The plot for a virtual cavity can be obtained by multiplying by a factor n^4.

The $½^+ \leftrightarrow ½^-$ and $1^+ \leftrightarrow 0^-$ or $1^- \leftrightarrow 0^+$ atomic transitions are suitable for experimental investigation of the dipole toroidal moment contribution to the $E1$ emission, because there is no $M2$ component in such transitions.

A numerical example

Moments of transition

$$< f\,|\,\hat{Q}^{C}_{LM}\,|\,i >= \sqrt{\frac{4\pi}{2L+1}}\int d^3r\rho_{fi}(\vec{r})r^L\mathrm{Y}_{LM}(\vec{r})$$

$$< f\,|\,\widehat{Q}^{T}_{LM}\,|\,i >= \frac{i}{2}\sqrt{\frac{4\pi}{2L+1}}\sqrt{\frac{L}{2L+1}}\int d^3r\vec{j}_{fi}(\vec{r})r^{L+1}\left(\vec{\mathrm{Y}}^{L-1}_{LM}(\vec{r})+\frac{2}{2L+3}\sqrt{\frac{L}{L+1}}\vec{\mathrm{Y}}^{L+1}_{LM}(\vec{r})\right)$$

Current

$$\vec{j}_{fi}(\vec{r}) = \frac{1}{2m_e i}\left(\psi^*_f(\vec{r})\vec{\nabla}\psi_i(\vec{r})-\psi_i(\vec{r})\vec{\nabla}\psi^*_f(\vec{r})\right)+\frac{\mu}{eS}\left[\vec{\nabla}\times\left(\psi^*_f(\vec{r})\hat{\vec{S}}\psi_i(\vec{r})\right)\right]$$

Data of calculation for atomic *E1* transition in H: $2P_{1/2} \rightarrow 1S_{1/2}$

$$< f\,\|\,\hat{Q}^{C}_{L=1}\,\|\,i >= \frac{2^7\sqrt{2}}{3^5}a_B$$

$$< f\,\|\,\widehat{Q}^{T}_{L=1}\,\|\,i >= \frac{a_B}{m_e}\frac{2^8\sqrt{2}}{5\times 3^6}\left(1+\frac{15}{2\sqrt{2}}\right)$$

A numerical example

Energy ω = 10.2 eV

$$\int d^3r \vec{A}_{LM}^{E^*}(k,\vec{r})\cdot\vec{j}_{fi}(\vec{r}) \propto \frac{\omega}{k} < J_f \| \hat{Q}_L^C \| J_i > + k < J_f \| \hat{Q}_L^T \| J_i >$$

$$\frac{\omega}{k} < J_f \| \hat{Q}_L^C \| J_i > + k < J_f \| \hat{Q}_L^T \| J_i > \cong \frac{1}{n} 0.74 a_B + n \frac{\omega}{m_e} 0.63 a_B$$

where $\omega / m_e = 2\times10^{-5}$

Toroidal moments in spin-ordered crystals

Hans Schmid

University of Geneva

Workshop on Super-Toroidal Electrodynamics,
University of Southampton, 5 November 2004

Overview

- History of the magnetoelectric effect and the axio-polar (time-odd polar) vector
- Ferroics, Ferrotoroidics and Multiferroics
- Detection of toroidal moments in crystals
- Postulated effects
  - Toroidic domains and toroidic walls
  - Electrotoroidic, magnetotoroidic and piezotoroidic effects
  - Toroidal optical SHG and toroidal optical rectification

History of the magnetoelectric (ME) effect

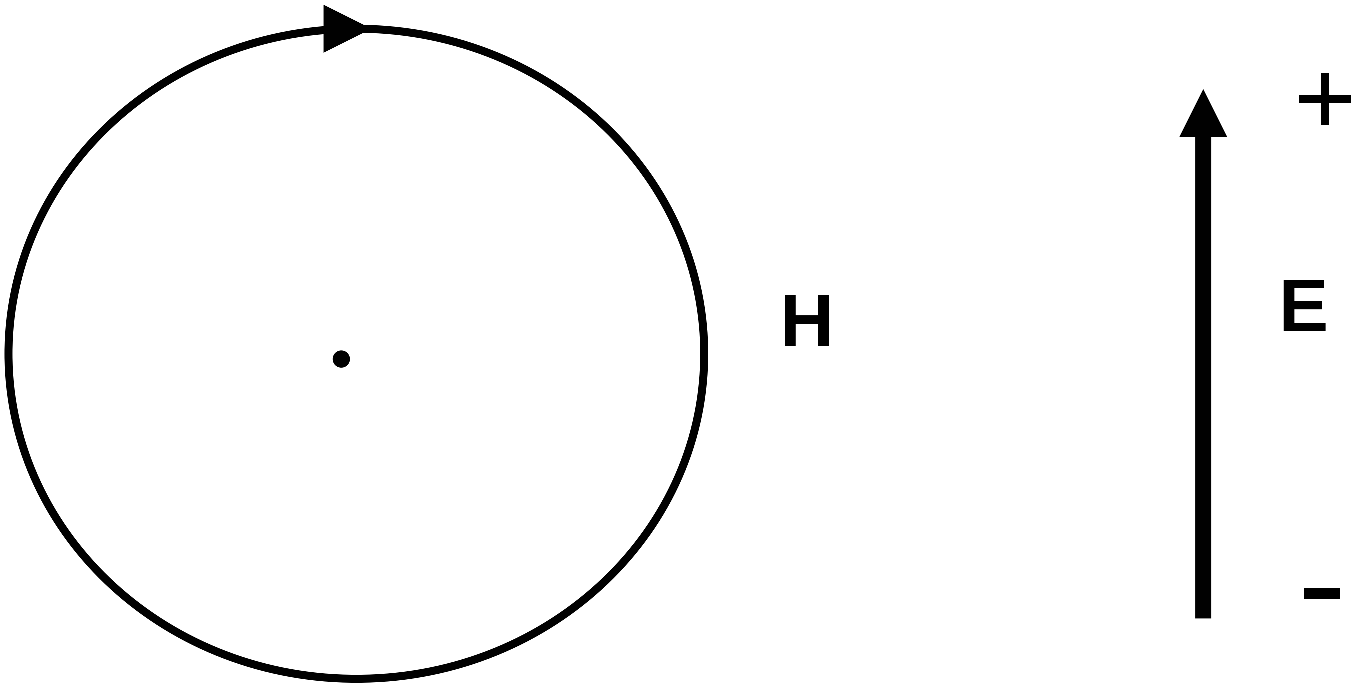
- **1894 Pierre Curie's conjecture:**

"Materials should exist, which can be polarised by a magnetic field and magnetised by an electric field "

Many unsuccessful experiments followed between 1922 and 1937 !

See: T.H. O'Dell, *The Electrodynamics of Magneto-electric Media*, North Holland, Amstedam,1970

1894 Pierre Curie knows the symmetry of the magnetic and electric field



History of the magnetoelectric (ME) effect

1932 Eugène **Wigner** introduces the **"time reversal"** symmetry operator R (= 1'):

- **For space inversion** $\bar{1}$: $\bar{1}\mathbf{E} = -\mathbf{E}$; $\bar{1}\mathbf{H} = \mathbf{H}$
- **Change of sign by applying** R:

velocity	R $\mathbf{v} = -\mathbf{v}$
electrical current density	R $\mathbf{j} = -\mathbf{j}$
spin density	R $\mathbf{S} = -\mathbf{S}$
magnetic field	R $\mathbf{H} = -\mathbf{H}$

- **No change of sign by applying** R:

charge density	R $\rho = \rho$
electric field	R $\mathbf{E} = \mathbf{E}$

History of the magnetoelectric (ME) effect

1937 Landau

nonmagnetic crystals **j = 0**

magnetic crystals R **j = - j =** $\neq$ **0**

R **S = - S**

1956 Landau and Lifshitz in *The Electrodynamics of Continuous Media (in Russian)*

The magnetoelectric effect and the piezomagnetic effect (not explicitly denominated) "should exist in principle <u>for certain magnetic classes</u>"

POINT GROUPS AND SPACE GROUPS

Crystallo-physical phenomenology

Crystal structure coordinates

32
Point groups
(Crystal classes)
J.F.C. Hessel 1830,
A. Bravais 1848

translation $\Rightarrow$

230
Space groups
Fedorov 1890,
Schönflies 1891

« Time reversal » $\Downarrow$

« Time reversal » $\Downarrow$

122
Shubnikov-Heesch point groups
A.V. Shubnikov 1929
H. Heesch 1930,
B.A. Tavger and V.M. Zaitsev 1956,
A.V. Shubnikov and N.V. Belov 1964

translation $\Rightarrow$

1651
Shubnikov space groups
A.M. Zamorzaev 1953,
N.V. Belov, N.N.Neronova, and T.S.. Smirnova 1955

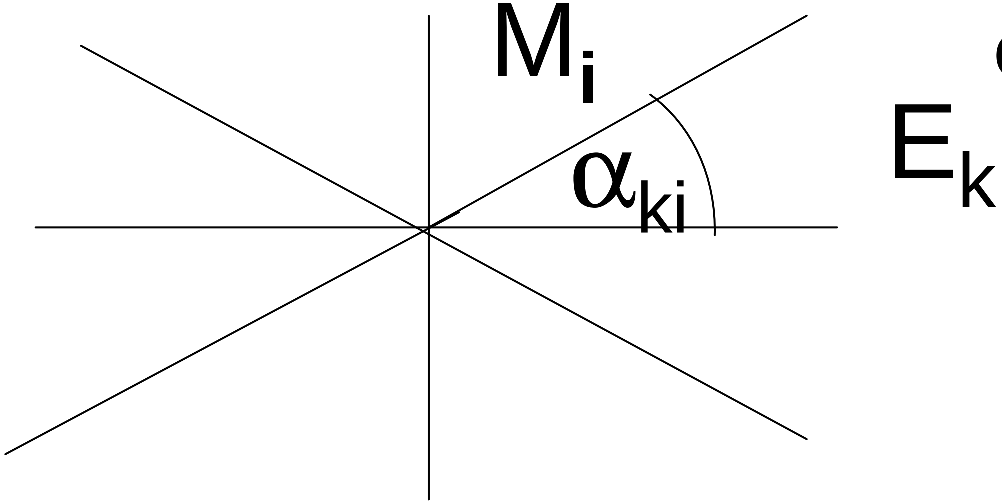
History of the magnetoelectric (ME) effect

- **1959** **Dzyaloshinsky** predicts the linear ME effect in a.f.m. Cr_2O_3 : **Point group $\bar{3}'m'$**

$$P_i = \alpha_{ik} H_k \quad \text{and} \quad M_k = \alpha_{ki} E_k$$

$$\boldsymbol{\alpha_{11} = \alpha_{22} \quad \alpha_{33}}$$

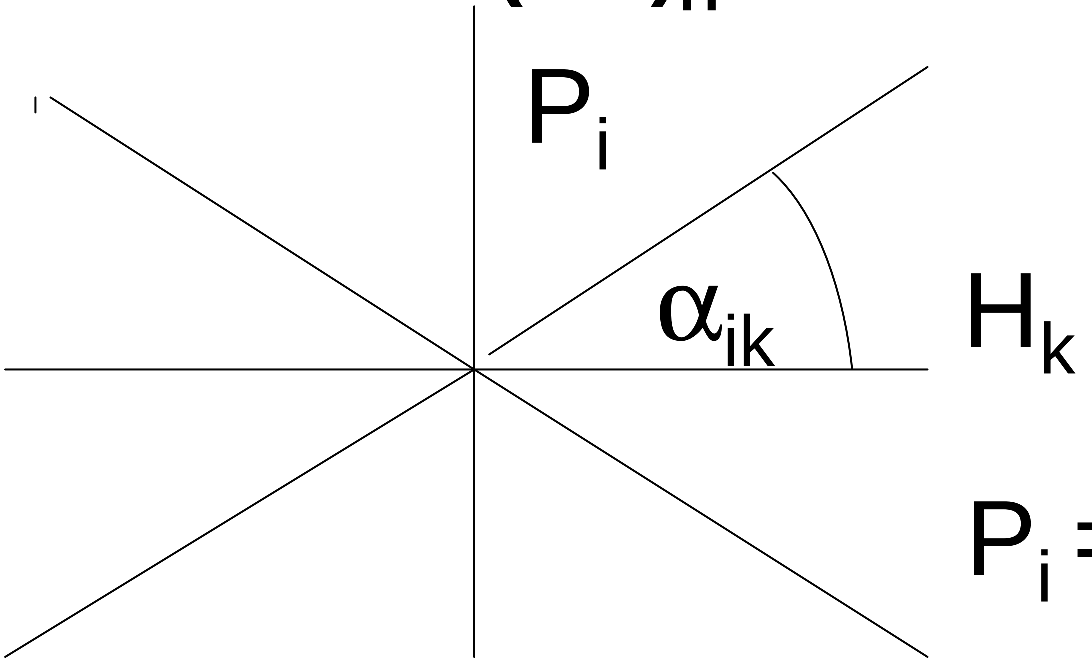
- **1960** **Astrov** measures the **$(ME)_E$** effect **on Cr_2O_3**



Dzyaloshinsky and Astrov, Ascona, September 1993

History of the magnetoelectric (ME) effect

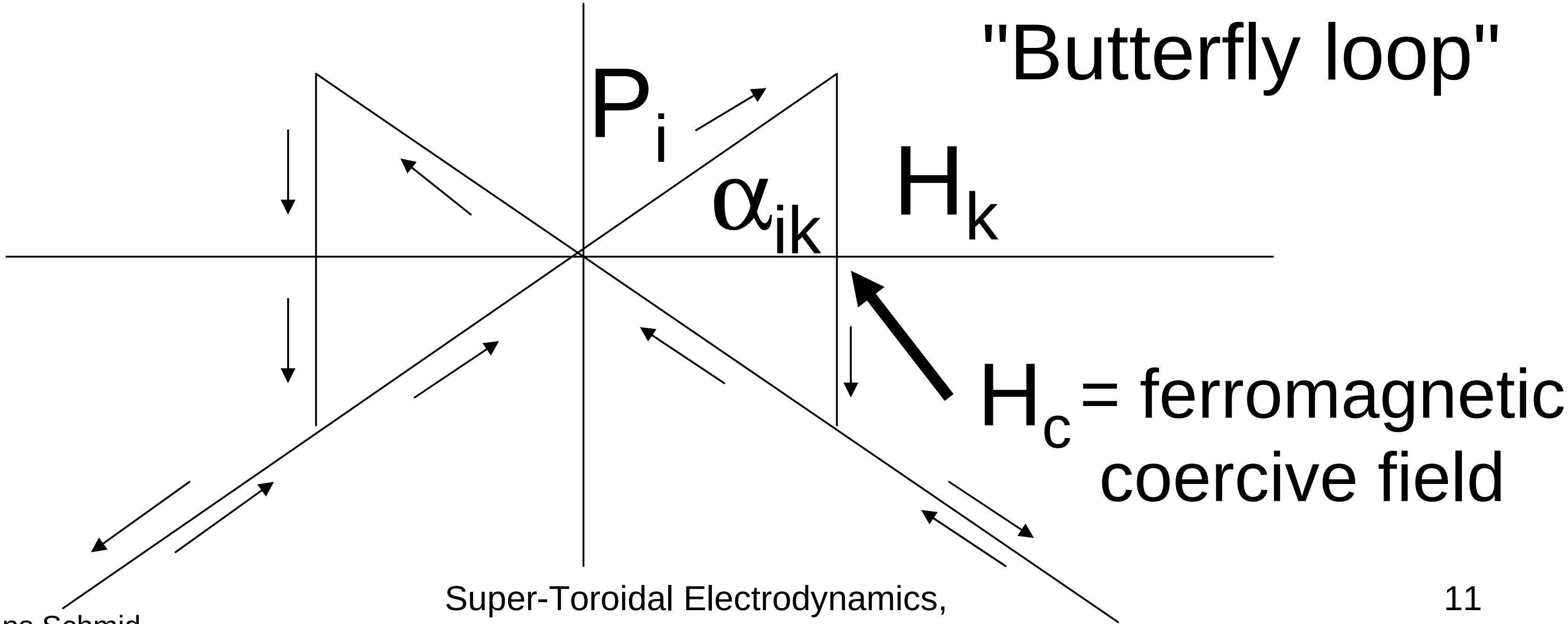
- **1961 Rado, Folen and Stalder** measure the $\mathbf{(ME)_H}$ effect on $\mathbf{Cr_2O_3}$ **:**



$$P_i = \alpha_{ik} H_k$$

History of the magnetoelectric (ME) effect

- **1966 (1964) Ascher, Rieder, Schmid and Stössel** measure the **$(ME)_H$** effect on ferroelectric/ferromagnetic $Ni_3B_7O_{13}I$



$$- g(\mathbf{E}, \mathbf{H}; T) = \ . \ . \ . + {}^{S}P_iE_i + {}^{S}J_iH_i + ½\ \varepsilon_0\varepsilon_{\mathrm{ik}}E_iE_k + ½\ \mu_0\mu_{\mathrm{ik}}H_iH_k$$
$$+ \alpha_{\mathrm{ik}}E_iH_k + ½\ \beta_{\mathrm{ijk}}E_iH_jH_k + ½\ \gamma_{\mathrm{ijk}}\ H_iE_jE_k + \ . \ . \ . \qquad (1),$$

where

$\varepsilon_0 = (1/(c^2\mu_0))$ [As/(Vm)] = free space permittivity — $\varepsilon_{\mathrm{ik}}$ relative permittivity

$\mu_0(=4\pi 10^{-7})$ [Vs/(Am)] = free space permeability — μ_{ik} relative permeability

c = free space light velocity ($\approx 3\ 10^8$[m/s])

${}^{S}\mathbf{P}$ [As/m^2] **=** spontaneous polarization

${}^{S}\mathbf{J}$ [Vs/m^2] = spontaneous magnetization

α [s/m] = tensor of linear ME effect, "*EH*" effect — α_{ik} non-symmetric in ik

β [s/A] = tensor of bilinear magnetoelectric"*EHH*" effect — β_{ijk} symmetric in jk

γ [s/V] = tensor of bilinear magnetoelectric"*HEE*" effect — γ_{ijk} symmetric in jk

 J.-P. Rivera, *Ferroelectrics*, **161**, 165-180 (1994)

Some magnetoelectric effects

- **Linear effects** **58 magnetic point groups**

$$P_i = \alpha_{ik} H_k \quad \text{and} \quad M_k = \alpha_{ki} E_k$$

- **Bilinear effects** **allowed in:**

$$P_k = \beta_{kij} H_i H_j$$ **66 piezoelectric point groups**

$$M_k = \gamma_{kij} E_k E_j$$ **66 piezomagnetic point groups**

Some magnetoelectric effects (contin.)

- **"Spontaneous effects" and "cross effects"** H. Schmid, *Ferroelectrics*, **221**, 9-17 (1999)

Switching (180° reversal) or reorientation (by angles other than 180°) of:

$^{s}P_i$ with E, H, σ

$^{s}M_i$ with E, H, σ

$^{s}T_i$ with E, H, σ, $S_i = (E \times H)_i$

$^{s}\varepsilon_{ij}$ with E, H, σ

Knowledge of prototype point group and of ferroic phase point group required !!

Species $\bar{4}3mFm'm2'$

H. Schmid,
Rost Kristallov
7, 32 (1967)
[*Growth of Crystals*, **7**, 25 (1969)]

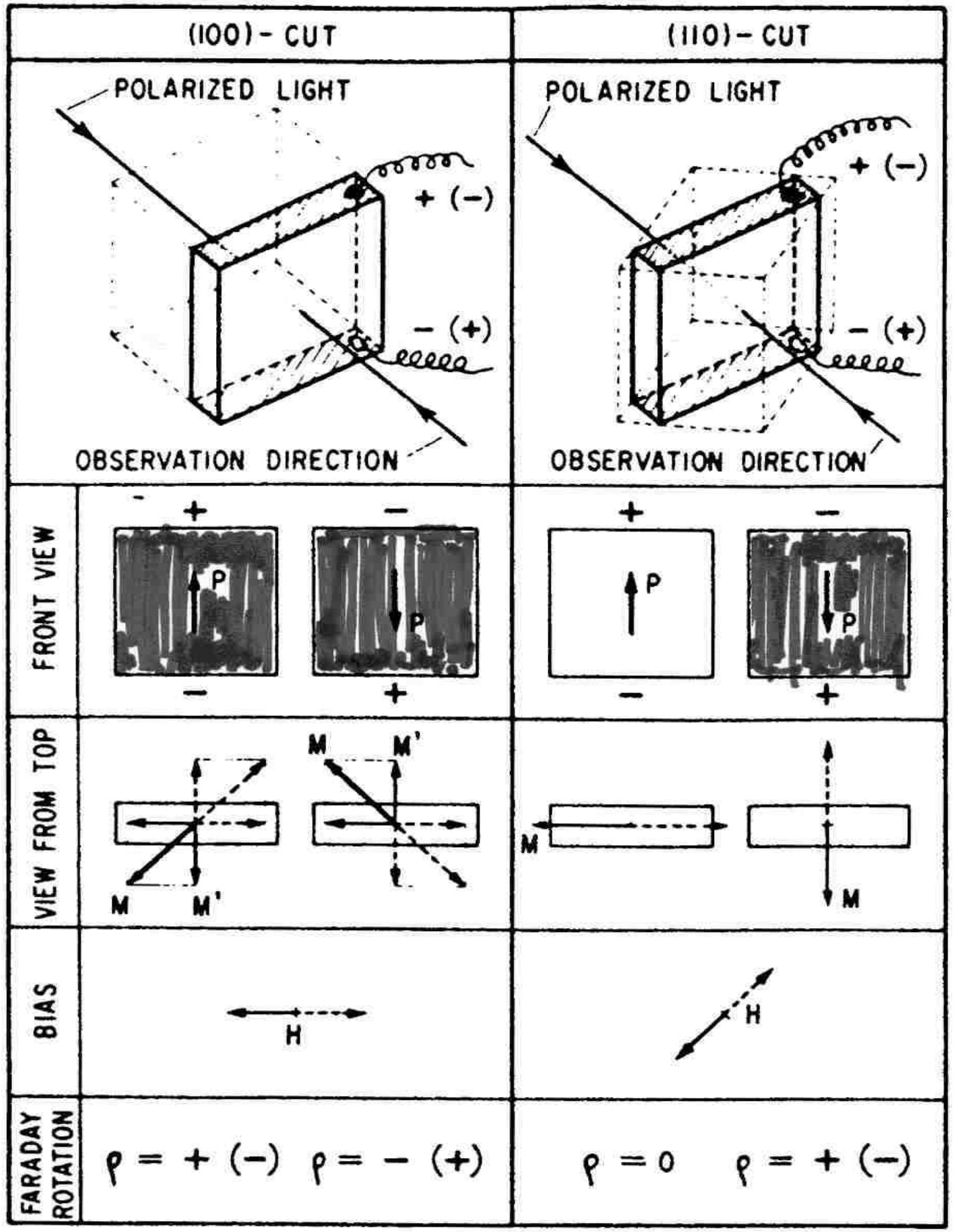


Fig. 25. Observation of electric and magnetic-field-induced domain switchings in NIB by means of the Faraday effect.

Supertoroidal Electrodynamics,
Southampton, 5 November 2004

Hans Schmid

THE AXIO-POLAR (TIME-ODD POLAR) VECTOR

- **1956 Lee and Yang**: propose that weak interactions destroy parity conservation
  T.D. Lee and C.N. Yang, *Phys.Rev*. **104**, 254-258 (1956)

- **1957 Wu et al**. demonstrate experimentally parity nonconservation in β-decay of cobalt-60.
  C.S. Wu, E. Ambler, R.W. Hayward, D.D. Hoppes and R.P. Hudson, *Phys.Rev.* **106**, 1361-1363 (1957)

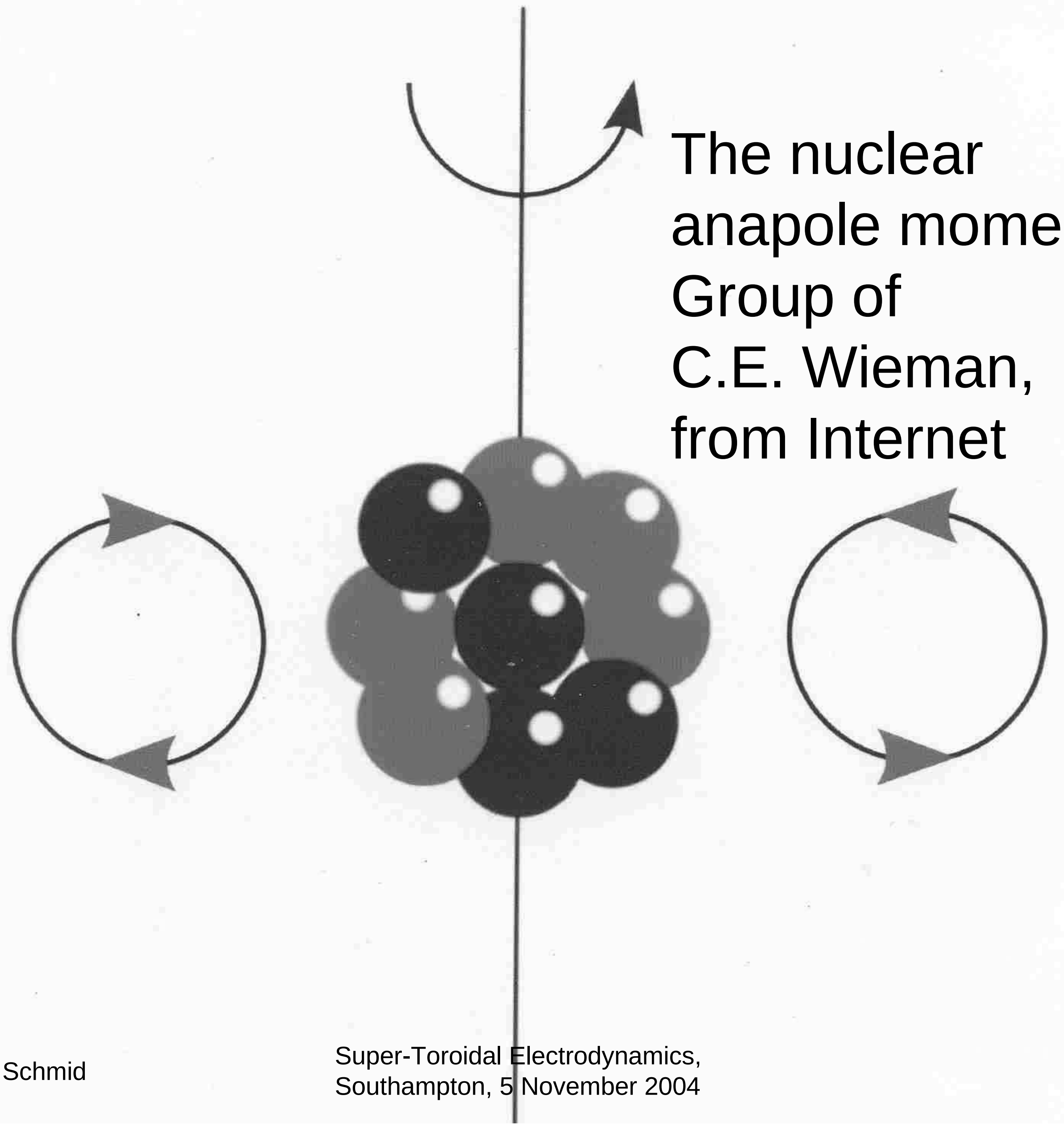
- **1957 Ya.B. Zel'dovich**
  Ya.B. Zeldovich, *Zh. Eksp. Teor. Fiz*., **33**, 1531 (1957) [*Sov. Phys. JETP*, **6**, 1184 (1958)]
  - Studies the origins of parity nonconservation
  - Finds that a system which has no definite parity generates a distribution of magnetic fields resembling a circular magnetic field of a toroidal winding: the **"anapole"** (name proposed by A.S. Kompanets). The anapole changes sign both under space and time reversal, hence a time-odd polar vector.

Science, **275**, 1759 (1997)

Measurement of Parity Nonconservation and an Anapole Moment in Cesium

C. S. Wood, S. C. Bennett, D. Cho,* B. P. Masterson,†
J. L. Roberts, C. E. Tanner,‡ C. E. Wieman§

The amplitude of the parity-nonconserving transition between the 6*S* and 7*S* states of cesium was precisely measured with the use of a spin-polarized atomic beam. This measurement gives $\mathrm{Im}(E1_{pnc})/\beta = -1.5935(56)$ millivolts per centimeter and provides an improved test of the standard model at low energy, including a value for the *S* parameter of $-1.3(3)_{exp}(11)_{theory}$. The nuclear spin–dependent contribution was 0.077(11) millivolts per centimeter; this contribution is a manifestation of parity violation in atomic nuclei and is a measurement of the long-sought anapole moment.



The nuclear anapole moment, Group of C.E. Wieman, from Internet

Two sources of parity non-conservation in atoms:

- Electron–nucleus weak interactions
- Magnetic interactions of electrons with the nuclear anapole moment

Independently of Zel'dovich
E. Ascher, *Helv. Phys. Acta*, 39, 40-48 (1966)

determines

- the 31 magnetic point groups permitting a "spontaneous current" # (i.e., in the absence of an external electric field)
- the 66 magnetic point groups permitting the "piezoconductive effect"
- The tensor form of the "piezoconductive effect" for all 66 groups

changing sign under **space and time** reversal,

hence it is an **axio-polar (time-odd polar) vector**

The four irreducible representations of the dihedral group $\bar{1}1'$ of order four, generated by space inversion $\bar{1}$ and time reversal $1'$

E	$\bar{1}$	$1'$	$\bar{1}1'$	Vector basis ***)	*), **)	Magnetic limiting point group***)
1	1	1	1	**G**	ρ, $\dot{\mathbf{M}}$, **grad P**, **n*****),	**∞/m1'**
1	- 1	1	- 1	**P**	**P**, $\dot{\mathbf{v}}$	**∞mm1'**
1	1	- 1	- 1	**M**	**M, grad v**	**∞/mm'm'**
1	- 1	- 1	1	**T**	**j, v, A**, $\dot{\mathbf{P}}$, **grad M**	**∞/m'mm**

*) E. Ascher, 1966 ; **) E.Ascher, 1973

***) V. Dubovik, S.S. Krotov, V.V. Tugushev, 1987

"SPONTANEOUS CURRENTS" and TOROIDAL MOMENTS

V.L. Ginzburg, A.A. Gorbatsevich, Yu.V. Kopaev and V.A. Volkov, *Solid State Commun*., **50**, 339-343 (1984)

- They give the 31 Shubnikov-Heesch point groups, **permitting a non-zero toroidal moment density.**
- "…one should bear in mind that **in a toroidal state a magnetoelectric effect must be observed…"** , i.e., the 31 groups must allow the linear magnetoelectric effect.

It turns out that **the 31 Shubnikov-Heesch point groups** given by Ginzburg et al. and permitting a toroidal moment, **are identical with the 31 point groups allowing an axio-polar (time-odd polar) vector**, e.g., velocity, current density, linear momentum, etc., as determined earlier by

E. Ascher, *Helv. Phys. Acta*, **39**, 40-48 (1966);
E. Ascher, *Int. J. Magnetism*, **5**, 287-295 (1973)

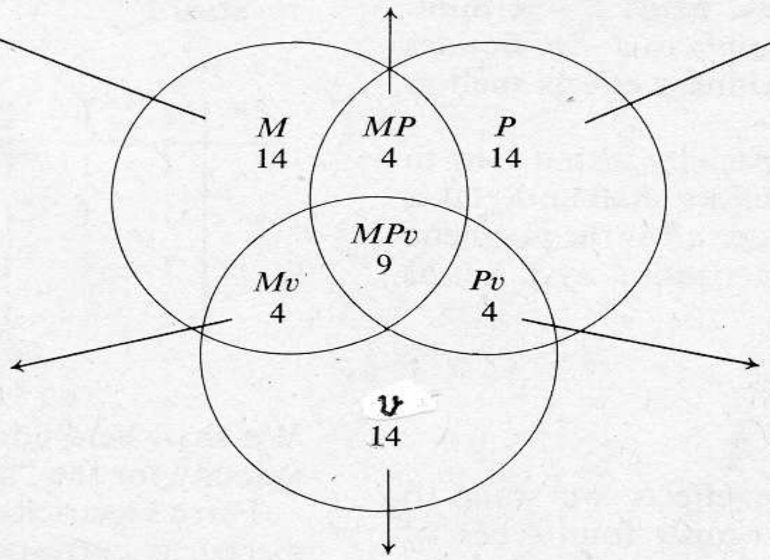
E. Ascher, *Int. J. Magnetism*, **5**, 287-295 (1974)

TABLE I

$\bar{1}$	$4/m$	$6/m$
$2/m$	$\bar{4}2'm'$	$\bar{3}m'$
$2'/m'$	$4/mm'm'$	$\bar{6}2'm'$
$mm'm'$	$\bar{6}$	$6/mm'm'$
$\bar{4}$	$\bar{3}$	

$2m'm'$, $4m'm'$, $3m'$, $6mm$

$1'$	$41'$	$61'$
$21'$	$4'mm'$	$3m1'$
$m1'$	$4mm1'$	$6'm'm$
$2mm1'$	$6'$	$6mm1'$
$4'$	$31'$	



$22'2'$, $42'2'$, $32'$, $62'2'$

$2mm$, $4mm$, $3m$, $6mm$

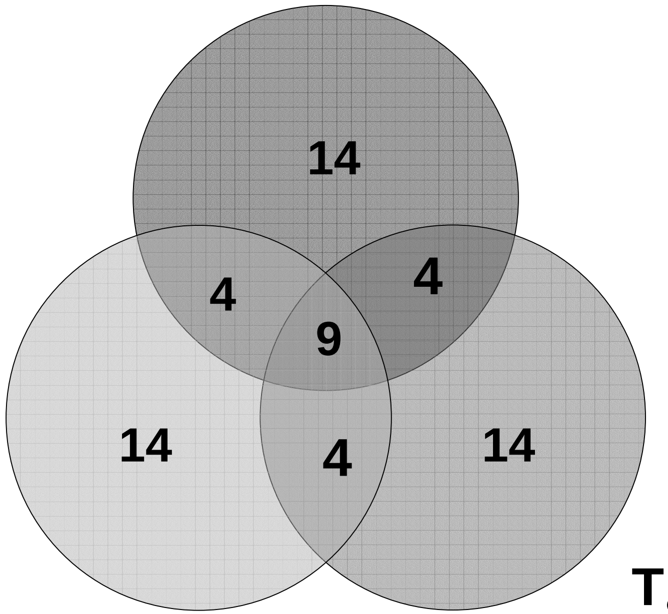
$\bar{1}'$	$4/m'$	$6/m'$
$2/m'$	$\bar{4}'m2'$	$\bar{3}'m$
$2'/m$	$4/m'mm$	$\bar{6}'m2'$
$m'mm$	$\bar{6}'$	$6/m'mm$
$\bar{4}'$	$\bar{3}'$	

MPv: 1, 2, m, $2'$, m', $mm'2'$, 4, 3, 6

The "Magic Trinity" of Groups

M_S
31 groups

Ferromagnetic



P_S
31 groups

T_S
31 groups

Ferrokinetic, Ferrotoroidic, Ferroconductive

Ferroelectric

E. Ascher, *Int. J. Magnetism*, **5**, 287-295 (1974)

SOME DEFINITIONS

- $\mathbf{p} = \int \mathbf{P}\, dV$ **electric dipole moment**
- $\mathbf{m} = \int \mathbf{M}\, dV$ **magnetic dipole moment**
- $\mathbf{t} = \int \mathbf{T}\, dV$ **toroidal dipole moment**
- $\mathbf{t} = \int \mathbf{T}\, dV = (1/10c) \int [(\boldsymbol{j}\cdot\boldsymbol{r})\boldsymbol{r} - 2r2\boldsymbol{j}]d3r$ (c.g.s.)
- where
- **P** **density of Polarization**
- **M** **density of Magnetization** (sometimes called “Magnetic moment”)
- **T** **density of Toroidal moment** (often referred to as “Toroidal moment”)
- ***J*** **current density**
- ***r*** **radius vector**
- c **free space light velocity**

V.L. Ginzburg,, *Usp fiz Nauk*, **171** (2001)., [*Physics-Uspekhi*, **44** (2001), 1037-1043/1041.]

V.L. Ginzburg, *Applications of Electrodynamics in Theoretical Physics and Astrophysics*, Gordon and Breach Science Publishers, New York, etc., 1989 (translation of 3rd Russian edition, 1987), p. 151

Definition of spin part of toroidal moment

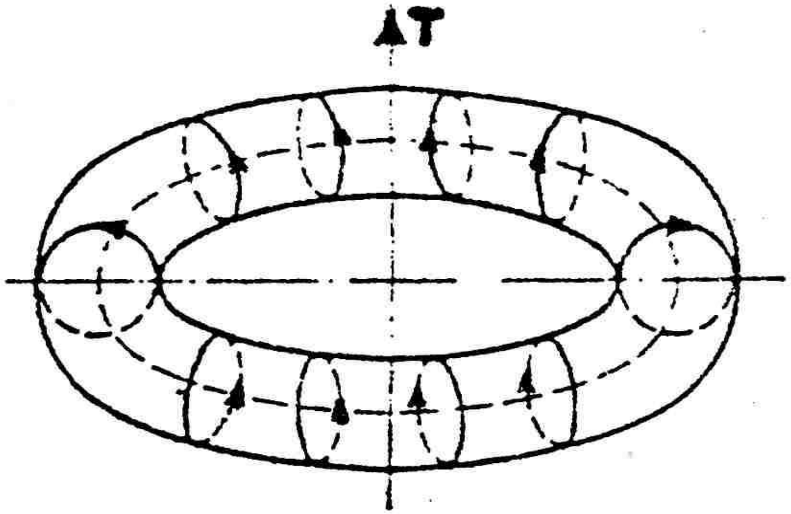
$$^{S}\mathbf{T} = \tfrac{1}{2}\,\mu_B \sum_a \mathbf{r}_a \times \mathbf{S}_a\,,$$

$\mathbf{S}_a$ = spin moment of magnetic cation « a »

$\mathbf{r_a}$ = radius vector of magnetic cation « a » from the unit cell's center

A.A. Gorbatsevich and Yu.V. Kopaev, *Ferroelectrics*, **161**, 321 (1994)

Toroidal moment of a solenoid formed into a torus *with an even number of windings*)*

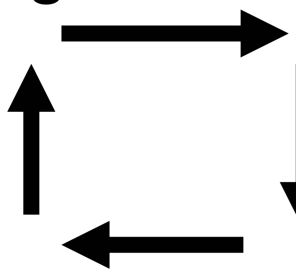


Magnetic limiting point group
∞/m' mm

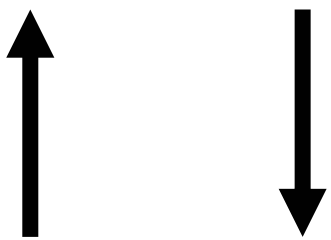
*) V. Dubovnik and L.A. Tosunyan, *Sov. J. Part. Nucl.*, **14**, 604 (1983)

Spin configurations with non-zero toroidal moment

- Head-to-tail spin configurations with four-fold axis **C_4** along the toroidal moment T



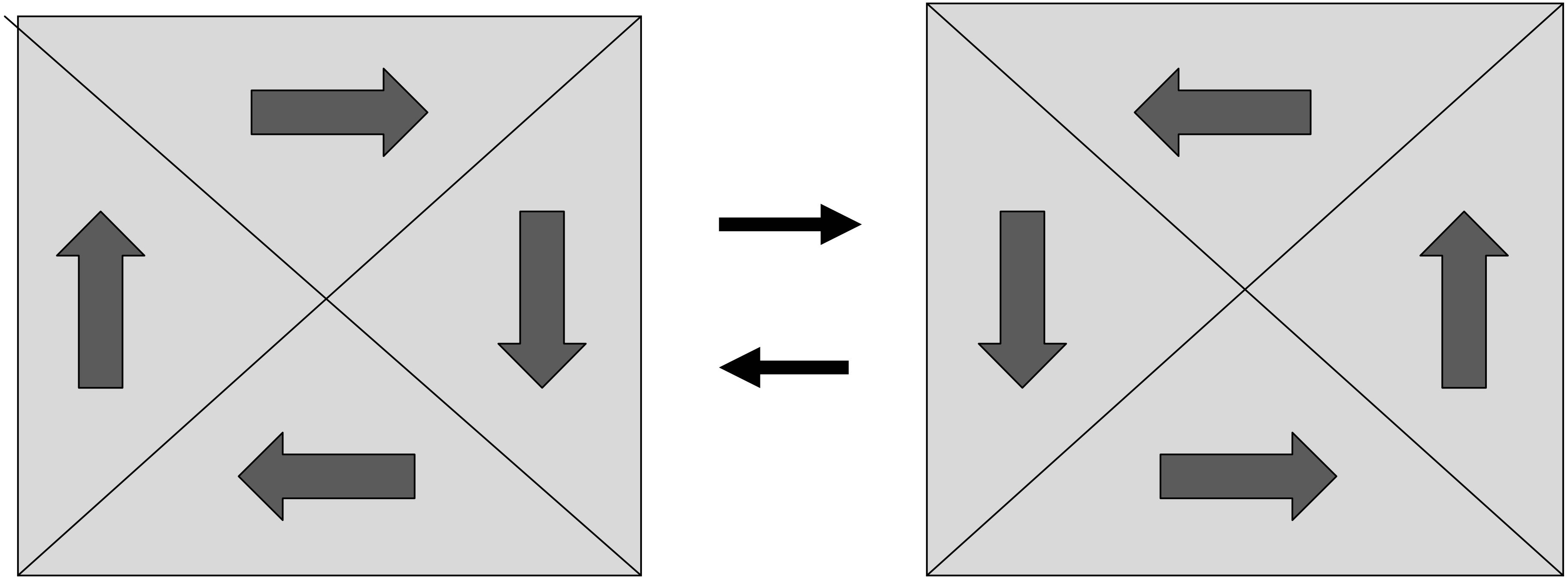
- Head-to-tail spin configuration with binary axis **C_2** along the toroidal moment T



A.A. Gorbatsevich and Yu.V. Kopaev, *Ferroelectrics*, **161**, 321-334 (1994)

Ferromagnetic domain configurations with non-zero toroidal moment

- Circular head-to-tail configuration of orthorhombic ferromagnetic domains
- Aizu-species 4/mmm1'/Fm'm'm(s)
- m'm'm does not allow a toroidal moment !!!



DOMAIN SWITCHING AND HYSTERESIS LOOPS

Toroidal moment contribution to stored free enthalpy

1) $\sim - \mathbf{T} \times \text{curl } \mathbf{H}$

A.A. Gorbatsevich and Yu.V. Kopaev, 1994

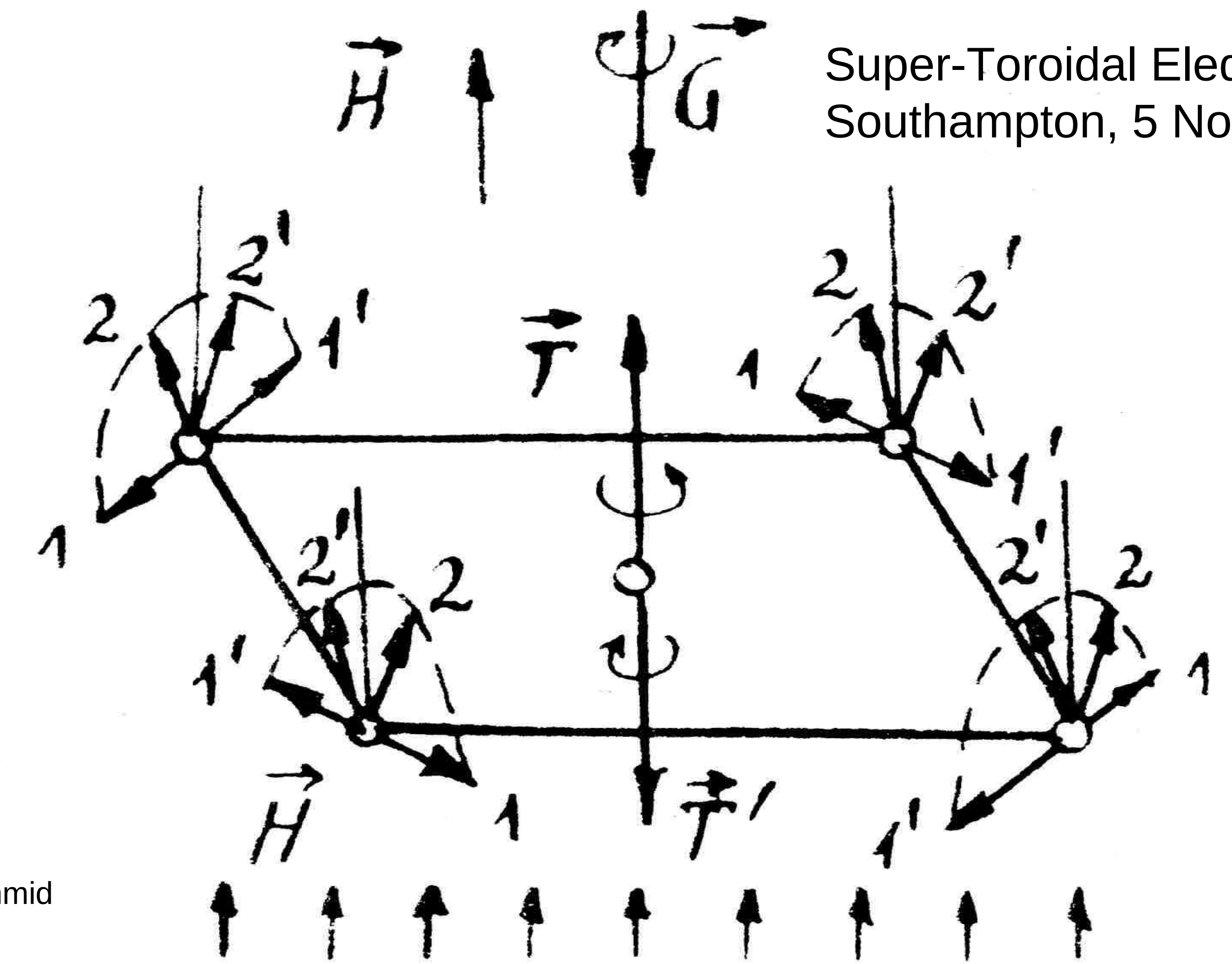
2) $\sim - \mathbf{T} S_i$, where $S_i = (E \times H)_i$

A.A. Gorbatsevich, Yu.V. Kopaev and V.V. Tugushev, 1983

V.M. Dubovik, M.A. Martsenyuk and N.M. Martsenyuk,
J. Magn. Magn. Mat., **145**, 211-230 (1995)

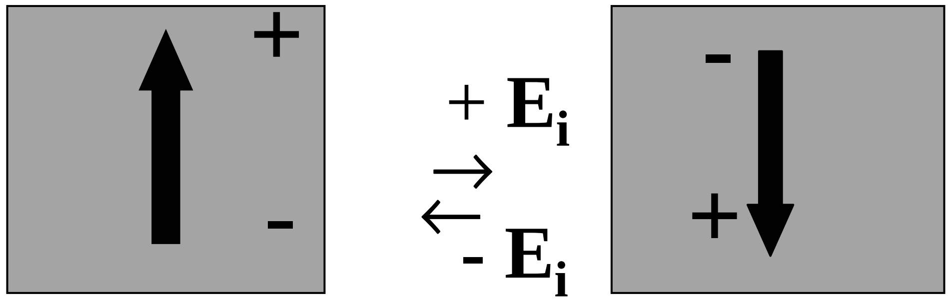
Proposed reversal of *T* by vorticity field *G* plus magnetic bias field *H, for* decreasing critical coercive field G_c

Super-Toroidal Eledtrodynamics,
Southampton, 5 November 2004

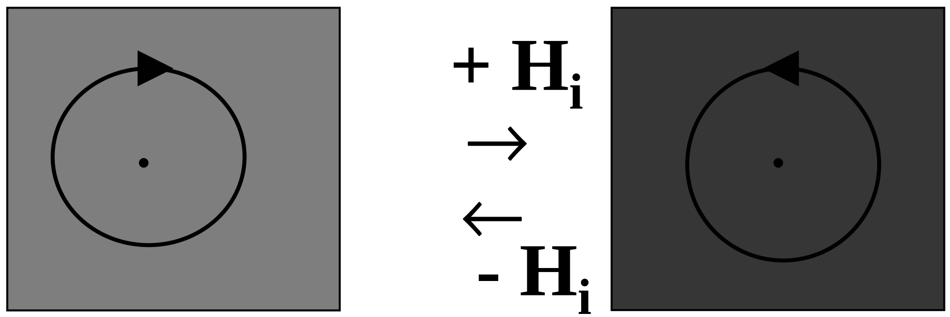


"FERROICS"

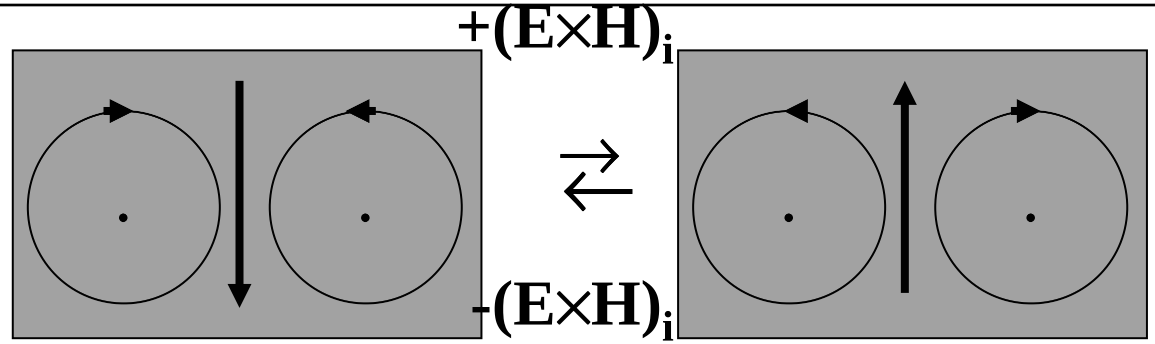
Ferroelectrics



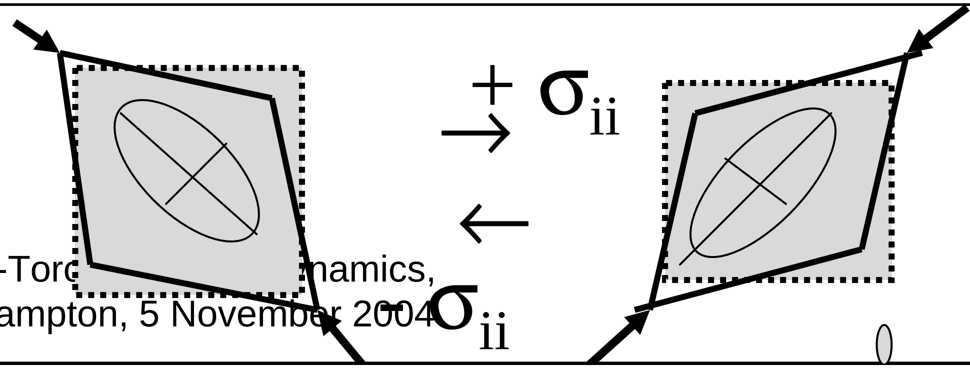
Ferromagnetics



Ferrotoroidics



Ferroelastics



H. Schmid, *Ferroelectrics*, **252**, 41-50 (2001)

Table X Ferroic "driving forces" of domain switching and reorientation due to differences in domain states (adapted from [1,2,3,15])

	"Driving force" $\Delta G \propto$	States differ in:	
Primary Ferroics			
Ferromagnetic	$\Delta {}^sM_i\, H_i$	spontaneous magnetization	${}^sM_i$
Ferroelectric	$\Delta {}^sP_i\, E_i$	spontaneous polarisation	${}^sP_i$
Ferrotoroidic	$\Delta\, {}^sT_i\, S_i$	spontaneous toroidal moment	${}^sT_i$
Ferroelastic	$\Delta {}^s\varepsilon_{ij}\, \sigma_{ij}$	spontaneous deformation	${}^s\varepsilon_{ij}$

$\Delta\, {}^sT_i(E \times H)_i$

Table X Ferroic "driving forces" of domain switching and reorientation due to differences in domain states (adapted from [1,2,3,15])

	"Driving force" $\Delta G \propto$	States differ in:	
Secondary ferroics			
Ferrobimagnetic	$\Delta\chi_{ij} H_i H_j$	magnetic susceptibility	χ_{ij}
Ferrobielectric	$\Delta\kappa_{ij} E_i E_j$	electric susceptibility	κ_{ij}
Ferrobielastic	$\Delta s_{ijkl} \sigma_{ij} \sigma_{kl}$	elastic compliance	s_{ijkl}
Ferroelastoelectric	$\Delta d_{ijk} E_i \sigma_{jk}$	piezoelectric coefficient	d_{ijk}
Ferromagnetoelastic	$\Delta q_{ijk} H_i \sigma_{jk}$	piezomagnetic coefficient	q_{ijk}
Ferromagnetoelectric	$\Delta\alpha_{ij} E_i H_j$	magnetoelectric coefficient	α_{ij}

***) H. Schmid, *Ferroelectrics*, **252**, 41-50 (2001)

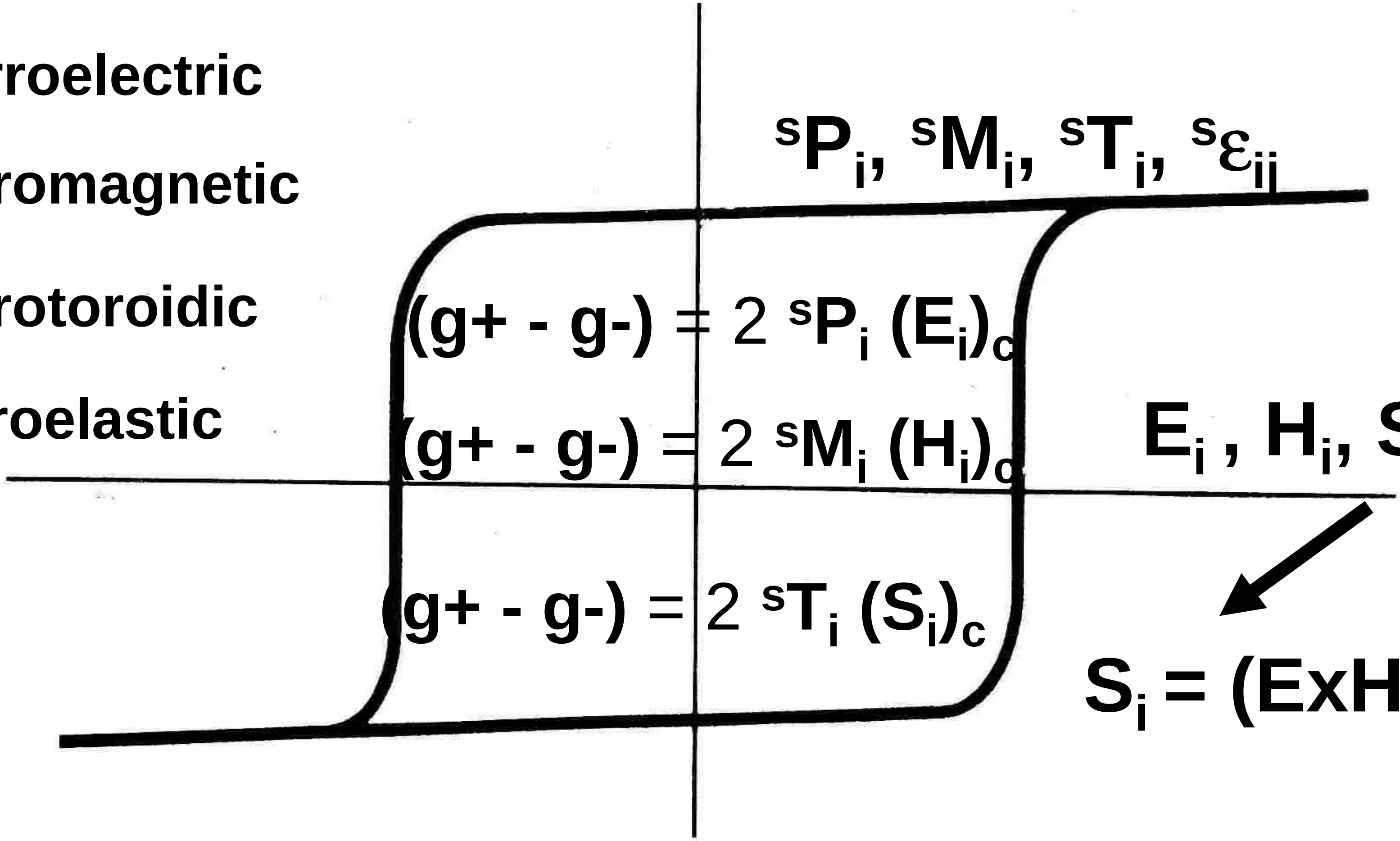
Primary ferroic hysteresis loops

ferroelectric

ferromagnetic

ferrotoroidic

ferroelastic



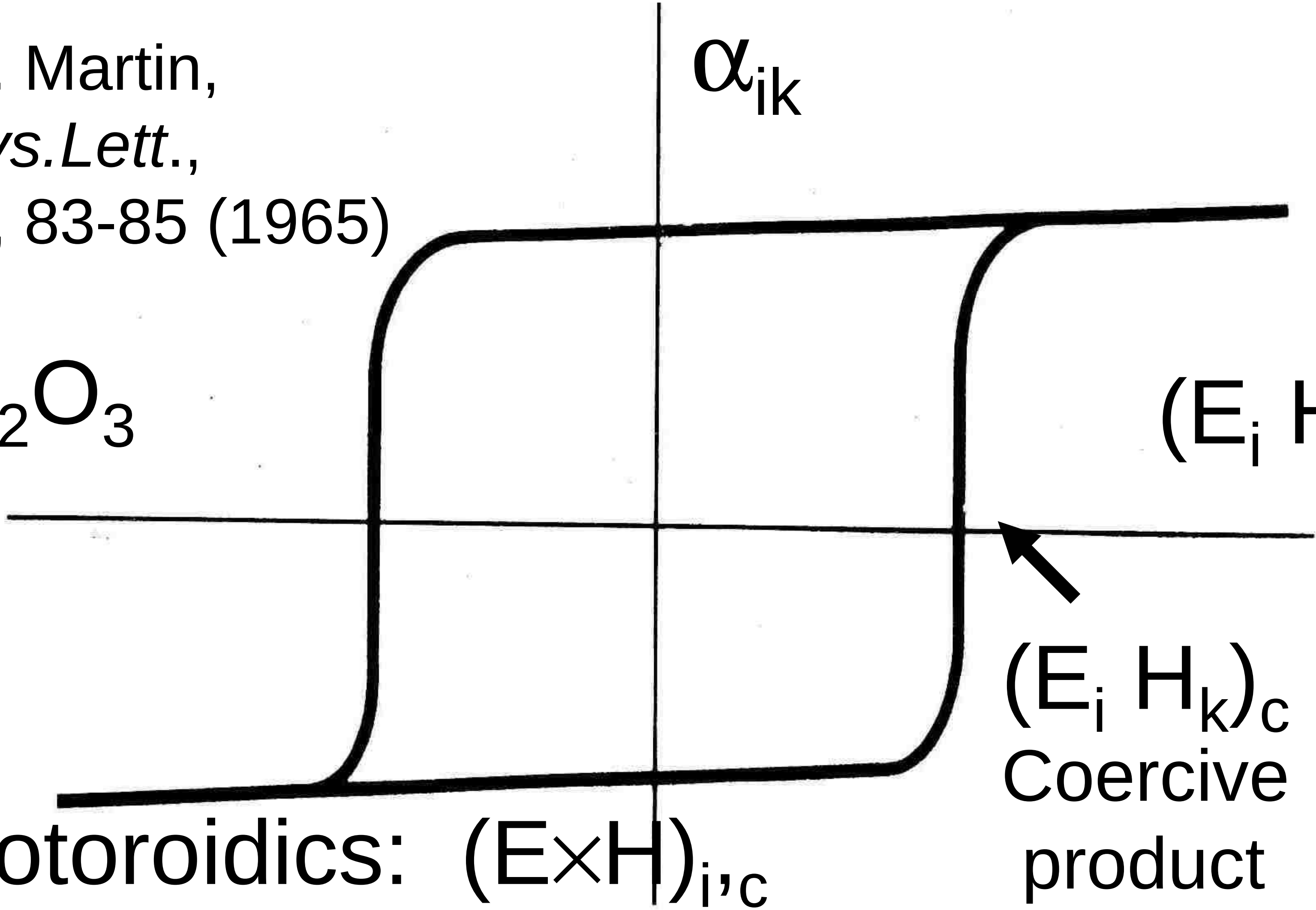
E_i , H_i, S_i, σ_{ij}

$S_i = (E \times H)_i$!!!

Secondary ferroic, magnetoelectric

T.J. Martin,
Phys.Lett.,
17, 83-85 (1965)

Cr_2O_3



Ferrotoroidics: $(E \times H)_{i,c}$

Magnetoelectric switching of antiferromagnetic domains of Cr_2O_3

(J.C. Martin and J.C. Anderson, 1966)

$$g^{+} = \tfrac{1}{2}\,\chi_{ik}\,H_iH_k + \tfrac{1}{2}\,\kappa_{ik}\,E_iE_k + \alpha_{ik}E_iH_k$$

$$g^{-} = \tfrac{1}{2}\,\chi_{ik}\,H_iH_k + \tfrac{1}{2}\,\kappa_{ik}\,E_iE_k - \alpha_{ik}E_iH_k$$

$g^{+} - g^{-} = 2\,\alpha_{ik}E_iH_k$ switching energy for the total hysteresis loop

Observed signatures of spontaneous toroidal moments in crystals

- Anomalous temperature dependence (singularities) of the linear ME effect (Boracites: $Ni_3B_7O_{13}I$, $Ni_3B_7O_{13}Cl$, $Co_3B_7O_{13}Br$, $Co_3B_7O_{13}I$)

- The asymmetry of the off-diagonal components of the linear ME effect tensor attests the presence of a toroidal moment in

  $Ga_{2-x}Fe_xO_3$

  Cr_2O_3 ($T_2 \sim \alpha_{31} - \alpha_{13}$)

  $BiFeO_3$ ($T_2 \sim \alpha_{21} - \alpha_{12}$)

D.E. Sannikov, *Ferroelectrics*, **219**, 177-181 (1989)

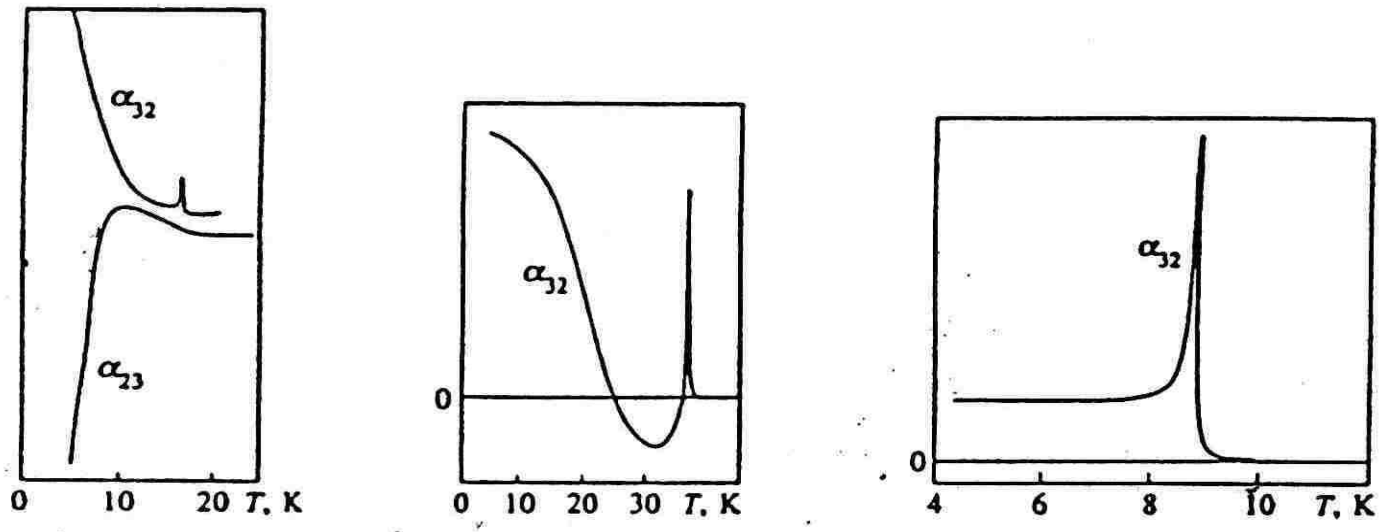


FIGURE 1 Temperature dependences of the components α_{32} and α_{23} of the magnetoelectric tensor in C_2 phase of Co–Br[1] (1), Co–I[2] (2), and Ni–Cl[3] (3) boracites.

Magnetic point group m'm2' : only coefficients α_{23} and α_{32}

$$\alpha_{32} = \frac{DaP_0}{xBC}\frac{1}{T_1} + \frac{1}{xB}\left(a + \frac{3D^2}{xC}a + \frac{3D}{C}b\right)T_1, \quad (4)$$

$$\alpha_{23} = -\frac{a}{\tilde{x}B}T_1, \quad (10)$$

A.A. Gorbatsevich, Yu.V. Kopaev and V.V Tugushev, 1983

identify

the physical meaning of the order parameter

T as *the antisymmetric component of the linear magneto-electric effect tensor*

N.B.: Any rank-2 tensor can be written as a sum of symmetric and antisymmetric parts as

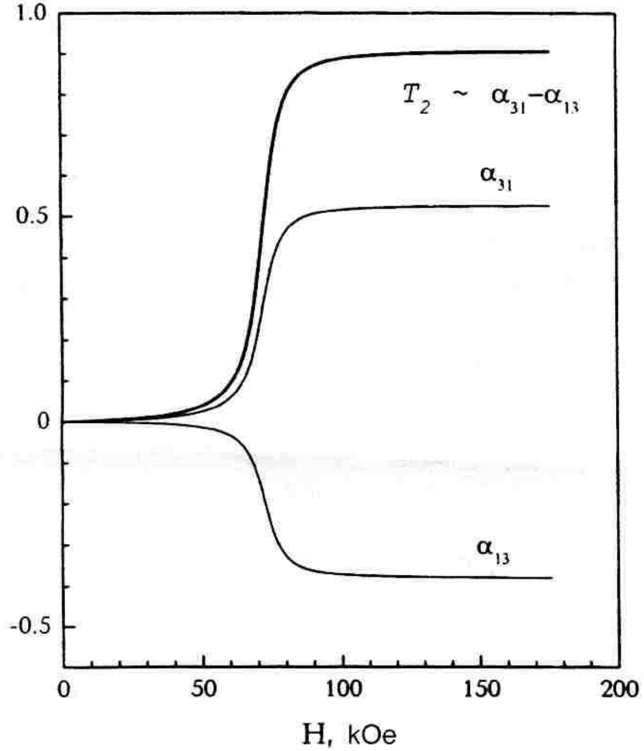
$$A^{mn} = \tfrac{1}{2}(A^{mn} + A^{nm}) + \tfrac{1}{2}(A^{mn} - A^{nm})$$

Magnetic field dependence of the off-diagonal components of the ME susceptibility tensor, α_{13}, α_{31}, and $T_2 \sim (\alpha_{31} - \alpha_{13})$ of the monoclinic (2'/m) toroidic spin-flop phase of Cr_2O_3 at 150K

Below H ~ 70 kOe Antiferromagnetic, magnetoelectric,

α_{11}, α_{11}, α_{33} non-toroidic,

$\bar{3}$'m'



Above ~ 70 kOe Antiferromagnetic, magnetoelectric,

α_{13}, α_{31}, α_{23}, α_{32} toroidic

2'/m

Yu. F. Popov, A.M. Kadomtseva, D.V. Belov, G.P. Vorob'ev and A.K. Zvezdin, *JETP Letters*, **69**, 330-335 (1999)

Postulated toroidic effects

- Electrotoroidic (toroido-electric) effect
- Magnetotoroidic (toroido-magnetic) effect
- Piezotoroidic (toroido-elastic) effect

- Toroidic optical SHG
- Toroidic optical rectification
- Toroidic domains and domain walls

E. Ascher, *Int. J. Magnetism*, **5**, 287-295 (1974)

Definition of the ferrokinetic, kineto-electric, kinetomagnetic and magneto-electric effects, respectively, with the terms of the density of stored free enthalpy g of the crystal

$$- g = \ldots + {}^{0}p_{i} \cdot v_i + \eta_{ik} v_i E_k + \zeta_{ik} v_i c B_k + \\ + \varepsilon_0 \lambda_{ik} E_i c B_k$$

E. Ascher, *Int. J. Magnetism*, **5**, 287-295 (1974)

$$-g\,(\boldsymbol{E},\boldsymbol{B},v) = \ldots + {}^{o}p.\,v + \eta_{ik}\,v_i E_k + \xi_{ik}\,v_i c B_k + \varepsilon_0\,\lambda_{ik}\,E_i\,c\,B_k$$

${}^{o}p$ = linear momentum without electric (E) and magnetic (B) fields

v = velocity

c = speed of light

η_{ik} = kineto-electric coefficient

ξ_{ik} = kinetomagnetic coefficient

λ_{ik} = magnetoelectric coefficient

Correspondence
Kinetic effects $\leftrightarrow$ Toroidic effects

E. Ascher, 1974 **H. Schmid, 2001**

- Density of free enthalpy:

$-g\,(\boldsymbol{E},\boldsymbol{B},\boldsymbol{v}) =$

$$\ldots + {}^{o}p_i . v_i + \eta_{ik} v_i E_k + \xi_{ik} v_i c B_k + \varepsilon_0 \lambda_{ik} E_i c B_k$$

$-g\,(\boldsymbol{E},\boldsymbol{H},\boldsymbol{S}) =$

$$\ldots + {}^{s}T_i S_i + \varphi_{ik} S_i E_k + \psi_{ik} S_i H_k + \alpha_{ik} E_i H_k$$

Primary ferroic

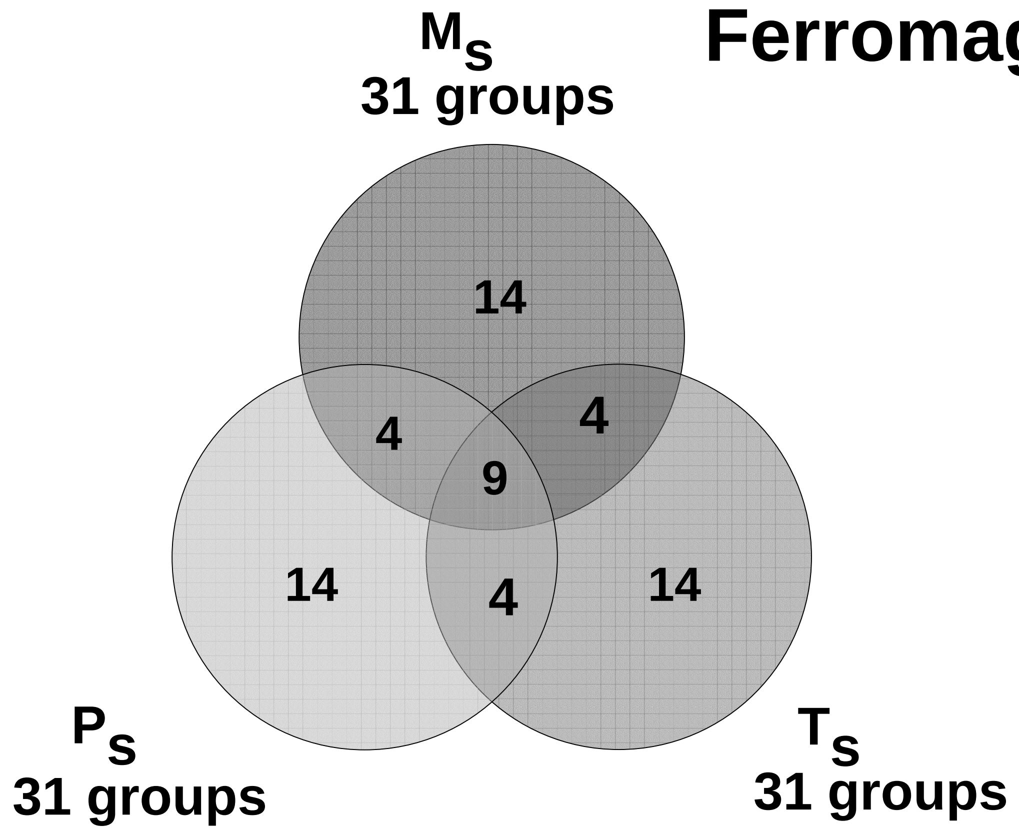
- **Ferrokinetic $\leftrightarrow$ Ferrotoroidic**

Secondary ferroics

- **Kineto-electric $\leftrightarrow$ toroido-electric**
- **Kinetomagnetic $\leftrightarrow$ toroidomagnetic**

The "Magic Trinity" of Groups

Ferromagnetic

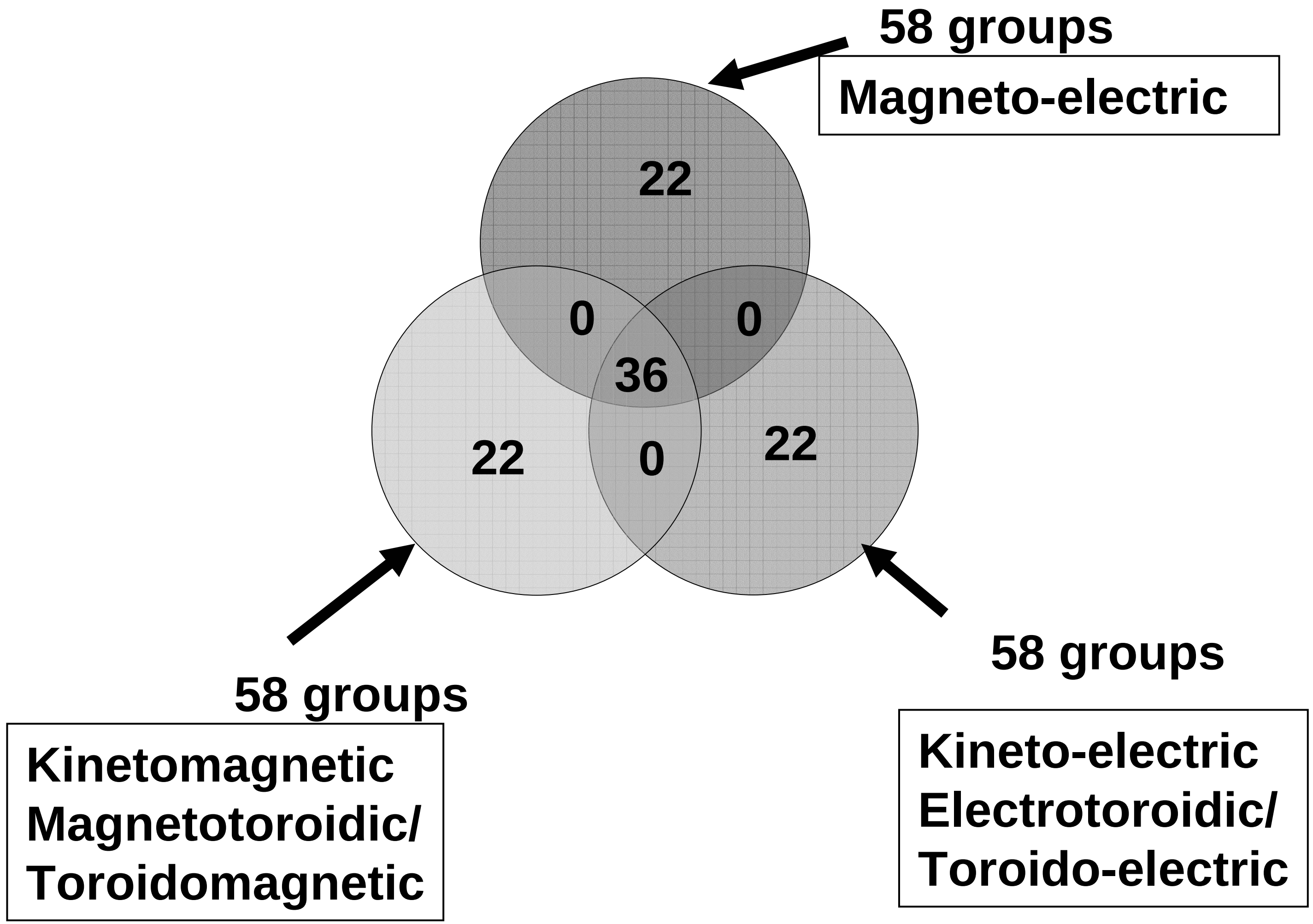


Ferroelectric

Ferrokinetic, Ferrotoroidic

E. Ascher, *Int. J. Magnetism*, **5**, 287-295 (1974)

The "Magic Trinity" of Groups



"Driving forces" (switching energy) for secondary ferroic domain switching $\propto$ terms of stored free enthalpy function

	E_j	H_j	S_j	σ_{kl}
E_i	EE	EH	ES	Eσ
H_i		HH	HS	H σ
S_i			SS	S σ
σ_{ij}				σ σ

Higher order magnetoelectric terms "hidden" in toroidal terms of stored free enthalpy function, with the restriction: $(E\times H)$

	E_j	H_j	S_j	σ_{kl}
E_i	EE	EH	$E(E\times H)$	$E\sigma$
H_i		HH	$H(E\times H)$	$H\sigma$
S_i			$(E\times H)(E\times H)$	$(E\times H)\sigma$
σ_{ij}				$\sigma\ \sigma$

DOMAINS

and

DOMAIN WALLS

Are toroidal domains and toroidal domain walls possible?

- D.E. Sannikov, Domain Wall in Ferrotoroic Phase of Boracites, *Ferroelectrics* **291**, 157-161 (2003)

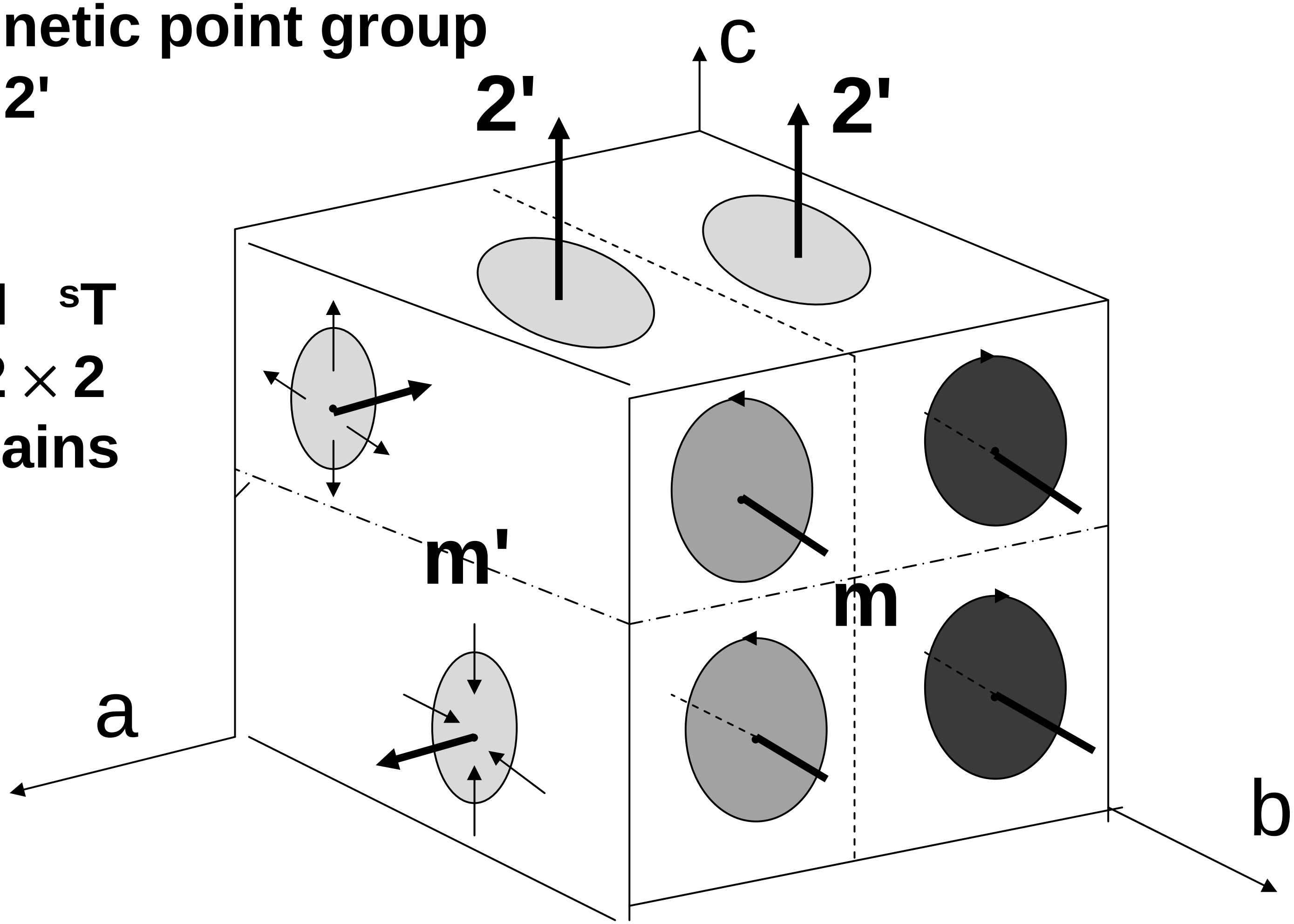
Magnetic point group m'm2'

- D.E. Sannikov, Dynamics of Domain wall in Ferrotoroic Phase of Boracites, *Ferroelectrics*, **291**, 163-168 (2003)

Toroidal domains ?

Magnetic point group
m'm2'

^{s}P ^{s}M ^{s}T
$1 \times 2 \times 2$
domains



How many domains ?

- **Perovskite P M T** $m\,\bar{3}\,m\,1'$

m'm2'	$12 \times 2 \times 2 = 48$
m	$24 \times 2 \times 2 = 96$
1	$48 \times 2 \times 2 = 192$ **!!**

- **Boracite P M T** $\bar{4}\,3\,m\,1'$

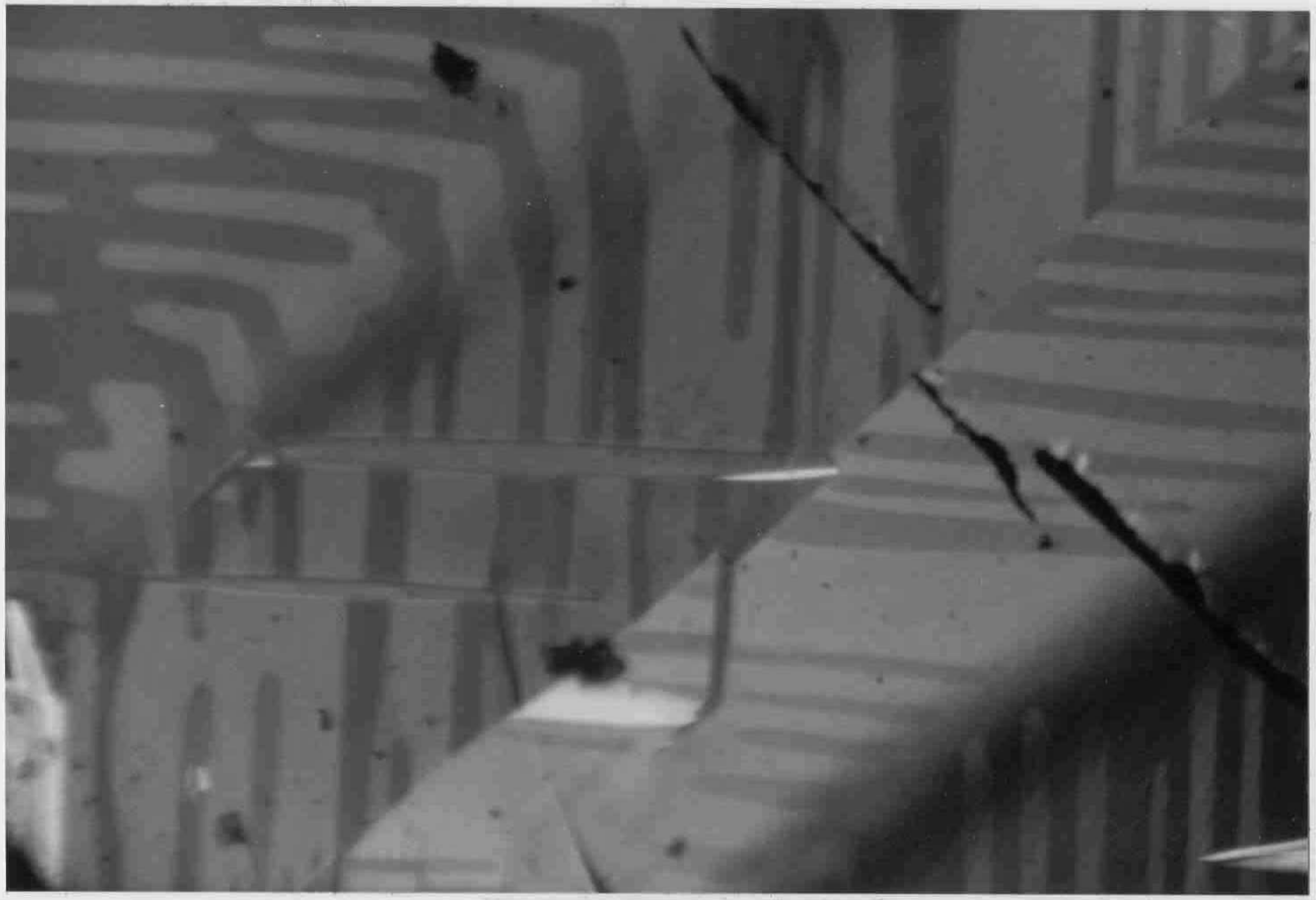
m'm2'	$6 \times 2 \times 2 = 24$
m	$12 \times 2 \times 2 = 48$
1	$24 \times 2 \times 2 = 96$ **!!**

How to make ferroic domains visible to the eye?

Domains	Method
• Ferroelastic	polarized light
• Ferroelectric=ferroelastic	polarized light
• Ferroelectric	etching, decoration, optical SHG, etc.
• Ferromagnetic	Faraday rotation, etc.
• Ferrotoroidic	**hopefully** non-linear optical spectroscopy (magnetic SHG topography ?) *)

*) See Manfred Fiebig and co-workers, Max-Bohr-Institut, Berlin:
http://mitarbeiter.mbi-berlin.de/fiebig/german/frame-mf.htm

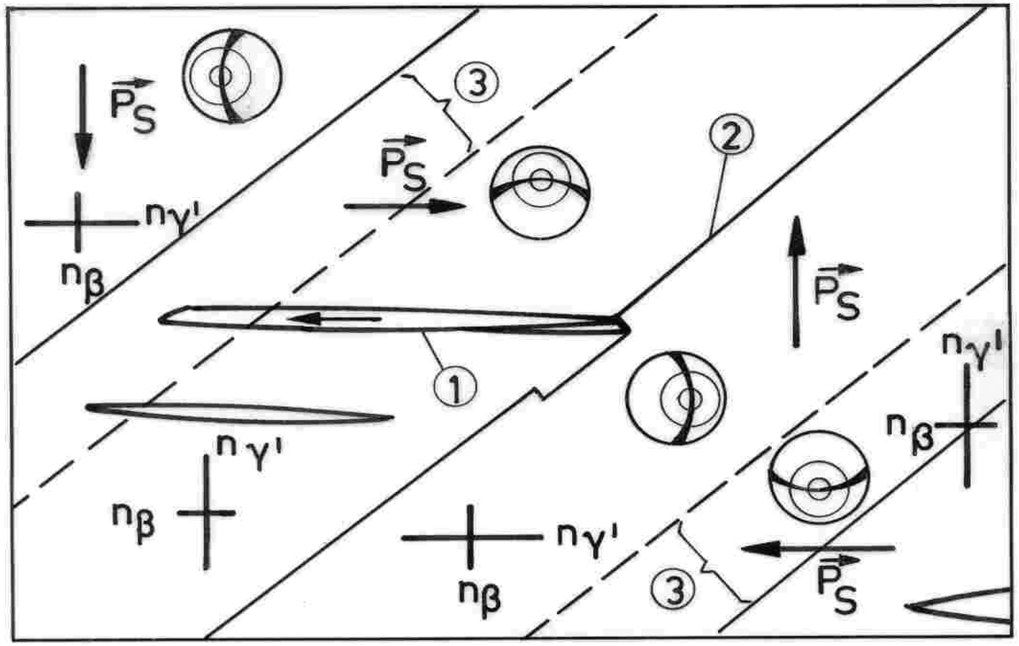
Nickel-iodine boracite $Ni_3B_7O_{13}Cl$, species $\bar{4}3mFm'm2'$,
cubic (100)-cut, ferroelectric/ferroelastic/ferromagnetic domains



T = 12K

Hans Schmid

Nickel-iodine boracite $Ni_3B_7O_{13}Cl$, species $\bar{4}3mFmm21'$, 20℃, cubic (100)-cut, ferroelectric/ferroelastic domains

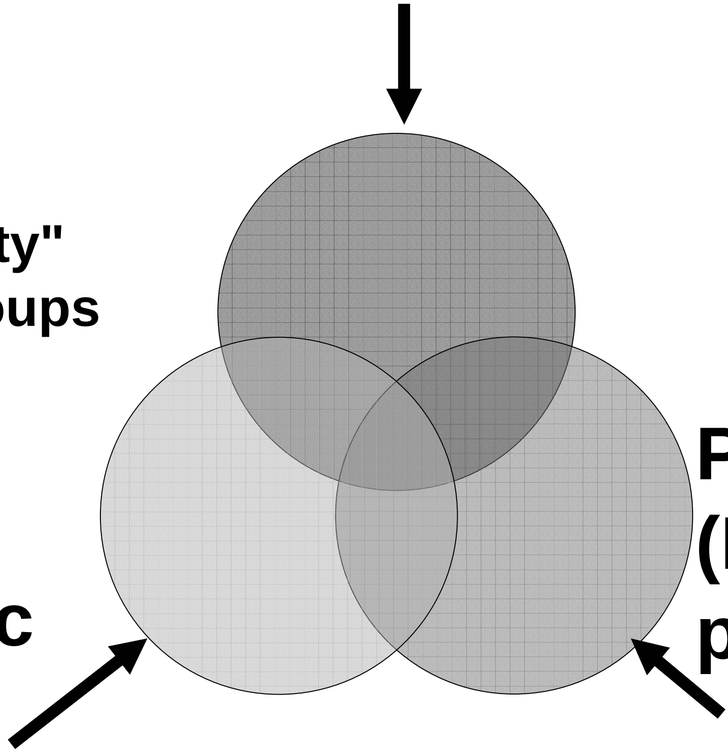


PIEZO-EFFECTS

Piezomagnetic

66 groups $H_i\sigma_{jk}$, $H_iE_jE_k$, $H_iH_jH_k$

The "magic trinity" of symmetry groups



Piezotoroidic (Piezoconductive, piezokinetic)

E. Ascher, 1966, 1974

Piezoelectric

66 groups

$E_i\sigma_{jk}$, $E_iE_jE_k$, $E_iH_jH_k$

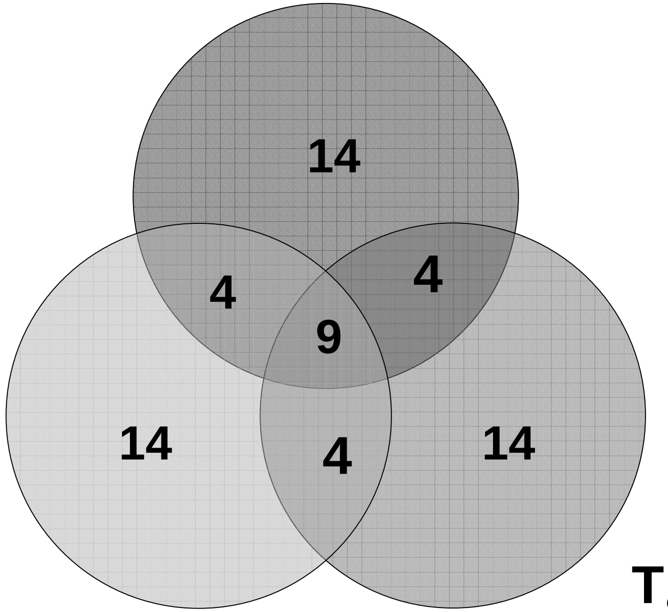
66 groups

$S_i\sigma_{ik}$, $S_iE_jE_k$, $E_iH_jH_k$

The "Magic Trinity" of Groups

M_S
31 groups

Ferromagnetic



P_S
31 groups

T_S
31 groups

Ferrokinetic, Ferrotoroidic, Ferroconductive

Ferroelectric

E. Ascher, *Int. J. Magnetism*, **5**, 287-295 (1974)

H. Schmid, *Ferroelectrics*, **252**, 41-50 (2001)

Table 3 Tensor form and type of transposed matrix form (Fig.1 of ref. [38]) of some secondary and tertiary ferroic terms of stored free enthalpy

Piezoelectric tensor form (t-type. matrix) :

$$E_i\sigma_{jk} \ , \quad E_iE_jE_k \ , \quad E_iH_jH_k$$

Piezomagnetic tensor form (s-type matrix):

$$H_i\sigma_{jk} \ , \quad H_iE_jE_k \ , \quad H_iH_jH_k$$

Piezotoroidic tensor form (u-type matrix) :

$$S_i\sigma_{jk} \ , \quad S_iE_jE_k \ , \quad S_iH_jH_k$$

[38] H. Grimmer, ***Ferroelectrics*, 161**, 181-189 (1994)

OPTICAL RECTIFICATION AND SHG

Optical second harmonic and optical rectification

$P_i^o = \frac{1}{2}\, \kappa_{ill}\, E_0^{\;2}$	Electric optical rectification
$P_i^{2\omega} = \frac{1}{2}\, \kappa_{ill}\, E_0^{\;2} \cos 2\omega t$	Electric optical 2nd harmonic
$M_l^0 \cdot = \frac{1}{2}\, \alpha_{iil}\, E_0^{\;2}$	Magnetic optical rectification
$M_l^{2\omega} = \frac{1}{2}\, \alpha_{iil}\, E_0^{\;2} \cos 2\omega t$	Magnetic optical 2nd harmonic
$T_l^0 = \frac{1}{2}\, \xi_{iil}\, E_0^{\;2}$	Toroidal optical rectification
$T_l^{2\omega} = \frac{1}{2}\, \xi_{iil}\, E_0^{\;2} \cos 2\omega t$	Toroidal optical 2nd harmonic

For the term *HHH* and Magnetic Optical Rectification see e.g. E. Ascher, *Helv.Phys.Acta,* **39**, (5), 466-476 (1966) / Appendix 3

For the « stored free enthalpy » function *g* see e.g. H.Schmid, *Int.J.Magnetism*, **4**, 337-361 (1973) / Table I

Conclusions

- **Toroidal moments in crystals are of magnetoeletric nature. Experimental evidence has been obtained**
- **Ferrotoroi(di)cs: a new kind of primary Ferroics $\rightarrow$ extension of "multiferroics"**
- **The microscopic theory and refined measurements of the (spontaneous) toroidal moment in crystals should be developed**

  Quantitative ME measurement of ${}^{s}T$, calculation of ${}^{s}T$ from nuclear and magnetic structural data, search for agreement of theory and experiment, orbital moments should be taken into account…

Conclusions (contin.)

- **Symmetry considerations allow to postulate some new toroidal effects:**

  - magneto-toroidic (toroido-magnetic) effect

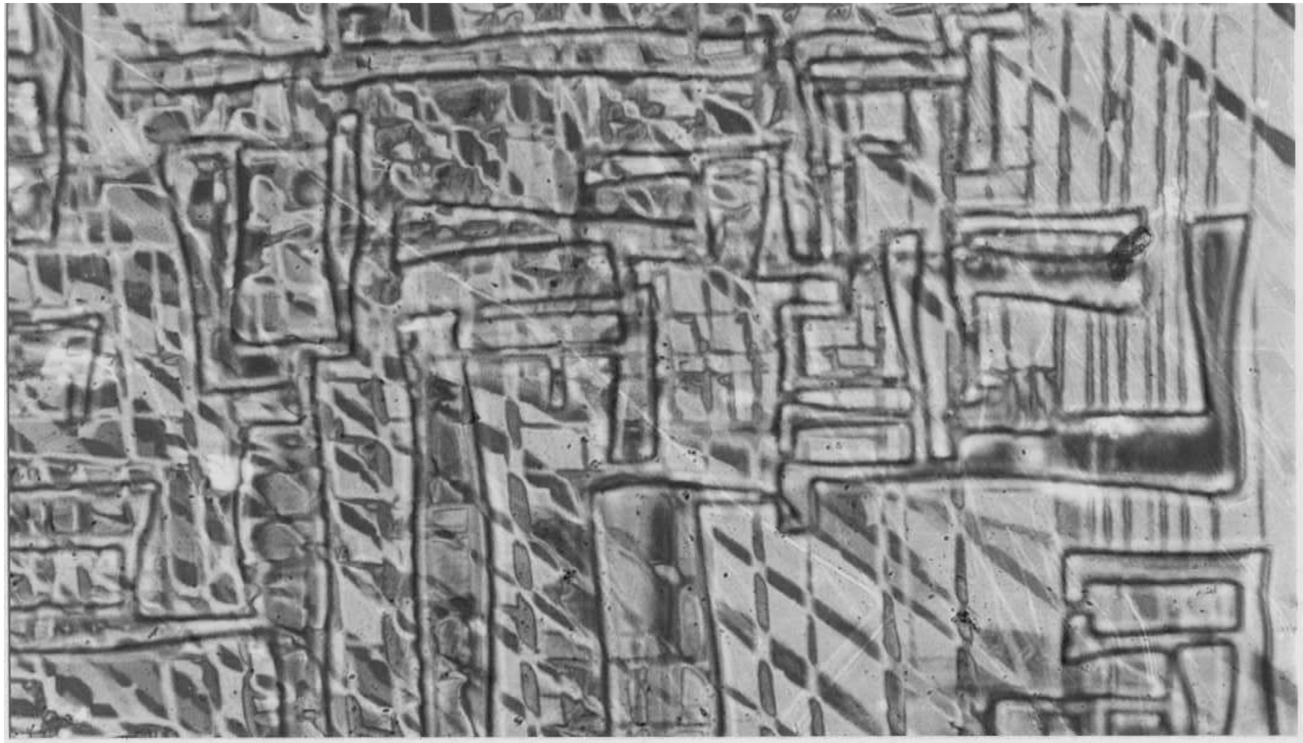
  - electro-toroidic (toroido-electric) effect

  - piezo-toroidic (toroido-elastic) effect

  - toroidic optical SHG and toroidic optical rectification

- **The existence of toroidal domains and domain walls is probable**
  - Desirable: attempts at revealing domains and walls by non-linear optical spectroscopy
  - Desirable: attempts at creating toroidic single domains by magnetoelectric/toroidic poling
- **Anti-toroidal magnetic structures may exist**

THANK YOU



FOR YOUR ATTENTION !

Toroidal electrodynamics and solid state physics

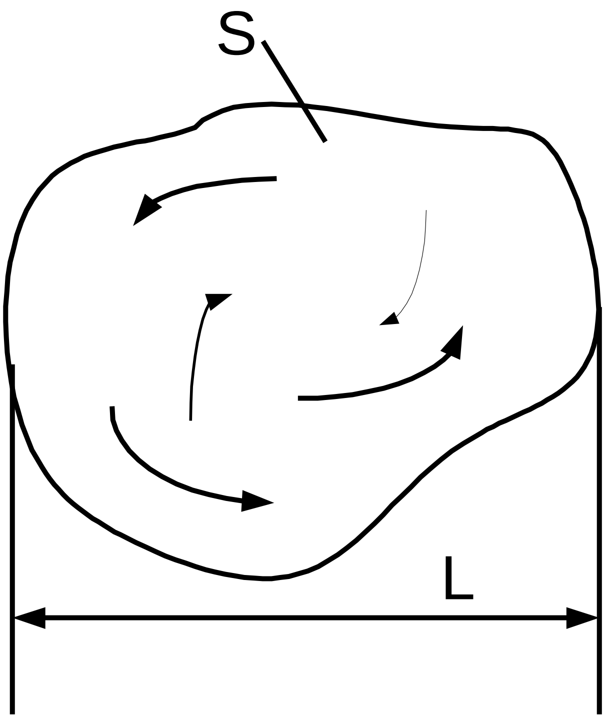
Mikhail Martseyuk

Perm State University (Russia)

Southampton University, 5 November 2004

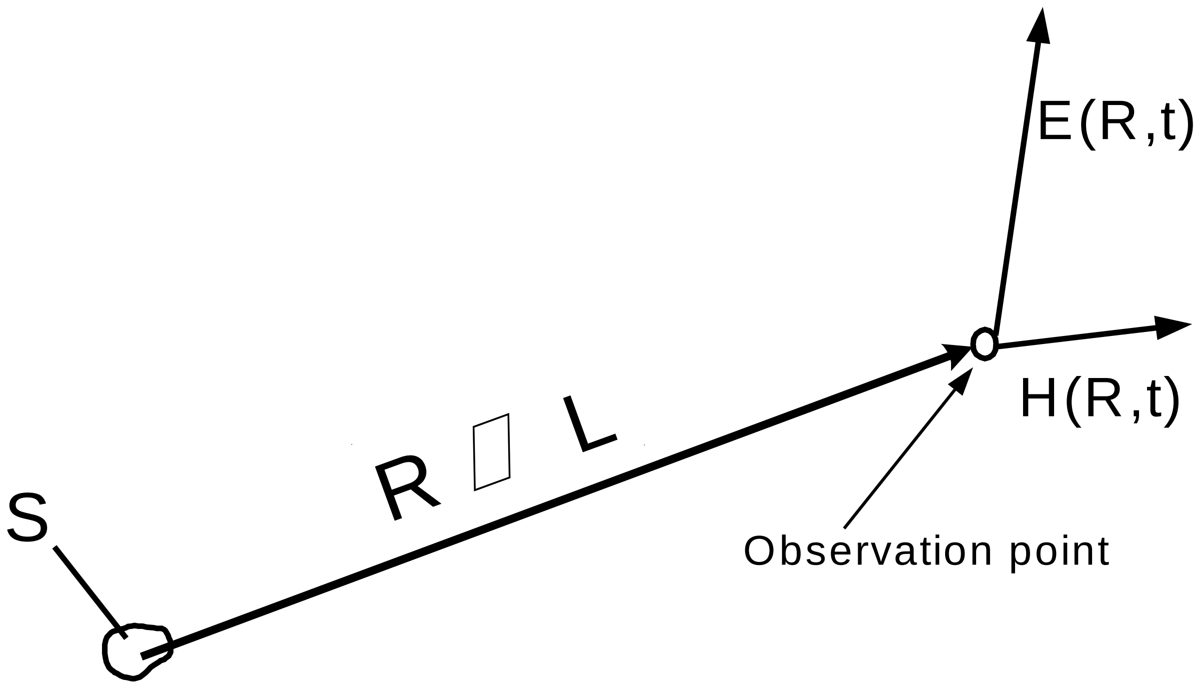
Toroid electrodynamics:

multipole decomposition of fields of finite system of moving charges



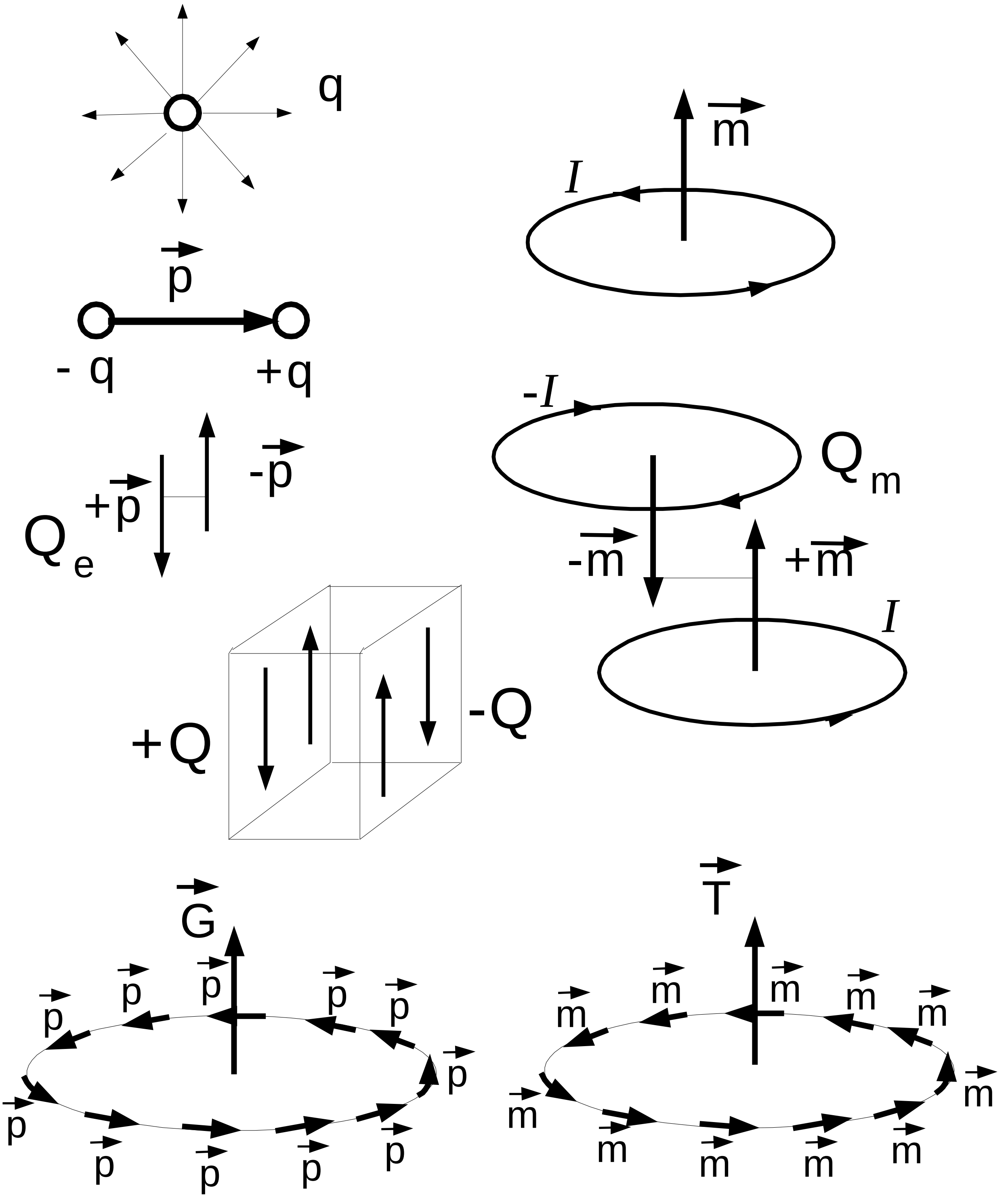
The system S of charges moving in finite region L is considered as a system of GIVEN quantities

$$\rho(\vec{r},t), \quad \vec{j}(\vec{r},t)$$



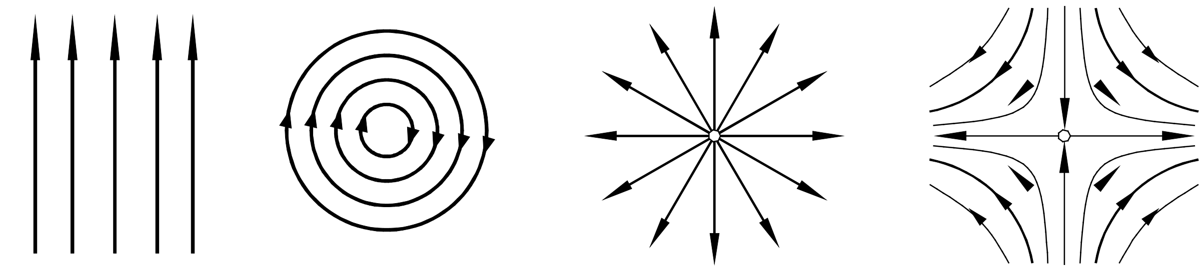
Our interest is to find electric and magnetic fields

Multipole moments



The system in external field

External fields classes of physical importance



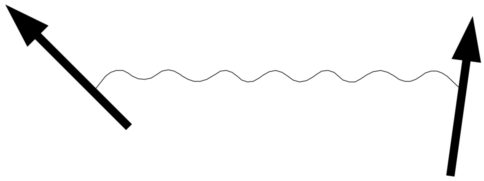
а) б) в) г)

Behavior of system in external field depends on the following:
– Does the system have fixed multipole moment or not?
– If not, what kind of polarization it is allowed by material?

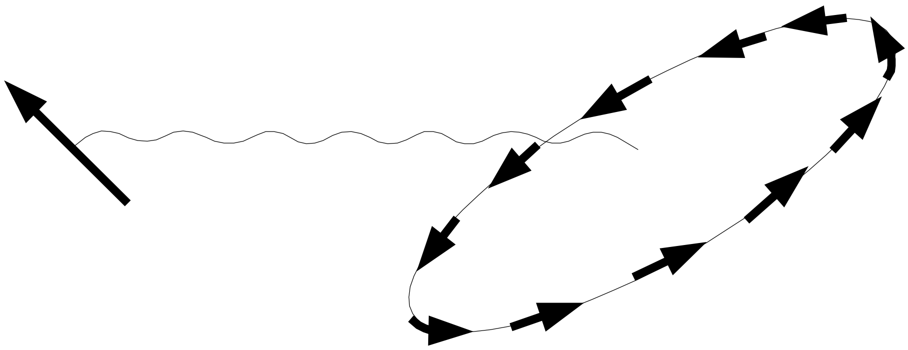
Examples of the systems with fixed multipole moment
– ferromagnetic particle (m)
– piroelectric particle (p)
– aggregate of ferromagnetic particles (T)
– antiferromagnetic particle (T)
– toroid memory cell made of ferromagnetic film (T)
– aromagnetic particle (G)

Examples of systems manifesting toroid polarization under the action of external field
– ferromagnetic fluid with aggregated particles (χ_T)
– suspension of antiferromagnetic particles (χ_T)
– suspension of aromagnetic particles (χ_G)
– nuclear (or effective atomic levels) spin system (χ_T)
– chiral molecule in uniform electric field (γ)

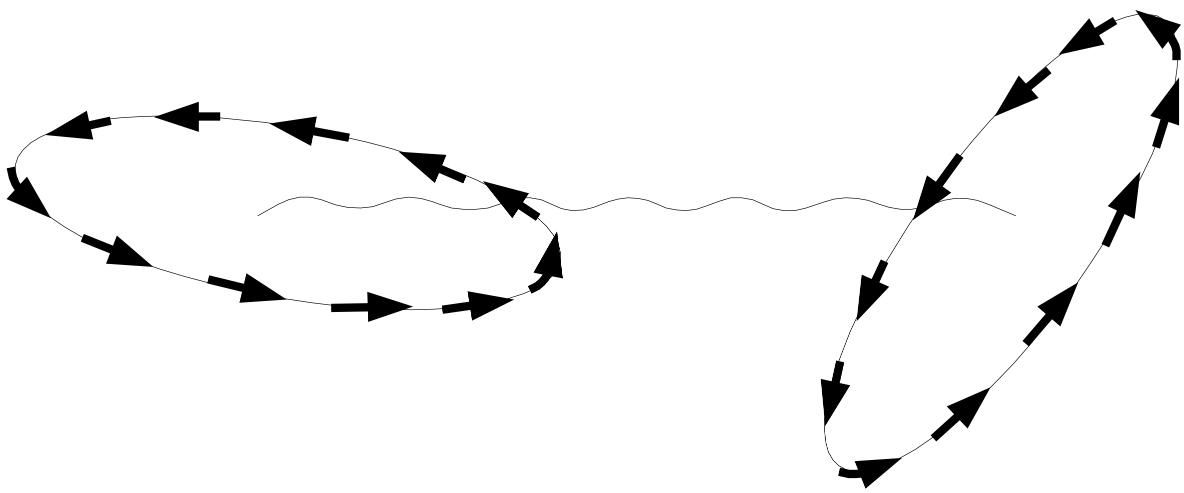
Interaction energy between multipoles



Dipole -dipole interaction

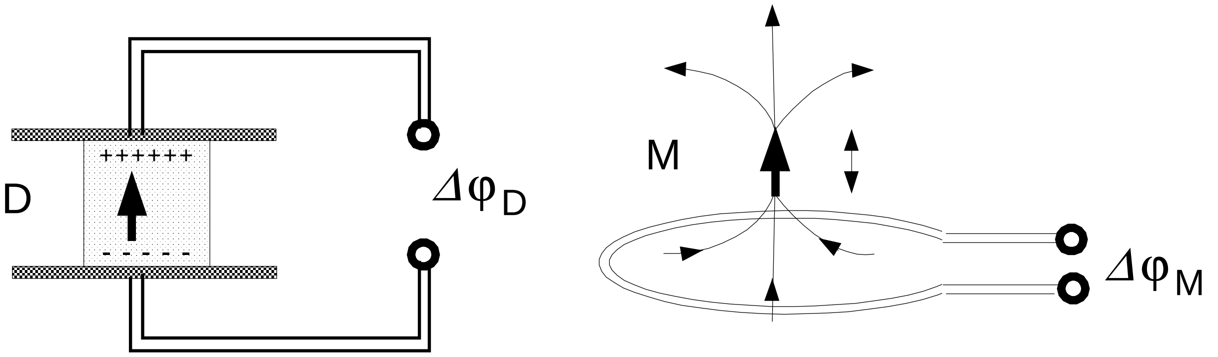


Dipole -toroid interaction



Toroid -toroid interaction

Direct measurement of multipole moments



	D	**M**
	$\mathbf{D}^{(eff)} = -\dot{\mathbf{T}}/c$	$\mathbf{M}^{(eff)} = \dot{\mathbf{G}}/c$
E	$\mathbf{D}\cdot Y^{(2)}$	$-\frac{1}{c}[\dot{\mathbf{M}}\mathbf{R}]/R^3$
	$-\frac{1}{c}\dot{\mathbf{T}}\cdot Y^{(2)}$	$-\frac{1}{c^2}[\ddot{\mathbf{G}}\mathbf{R}]/R^3$
H	$\frac{1}{c}[\dot{\mathbf{D}}\ \mathbf{R}]/R^3$	$\mathbf{M}\cdot Y^{(2)}$
	$-\frac{1}{c^2}[\ddot{\mathbf{T}}\ \mathbf{R}]/R^3$	$\frac{1}{c}\dot{\mathbf{G}}\cdot Y^{(2)}$

$$Y_{ik}^{(2)}(R) = \frac{3X_iX_k - R^2\delta_{ik}}{R^5}$$

Estimation of induced signal

Момент	Оценка	Разн. пот.	Измерит.система
D	$\sim eaNL^3$	$\sim 4\pi eaNL$	Электроды
T	$\sim ea\frac{v}{c}R_cN'L^3$	$\sim 4\pi ea\frac{vu}{c^2}N'R_c$	Электроды
M	$\sim ea\frac{v}{c}NL^3$	$\sim ea\frac{v}{c}NLu$	Виток
G	$eaR_cN'L^3$	$ea\frac{u^2}{c^2}N'R_c$	Виток

$$v = a\omega; \quad u = L\Omega.$$

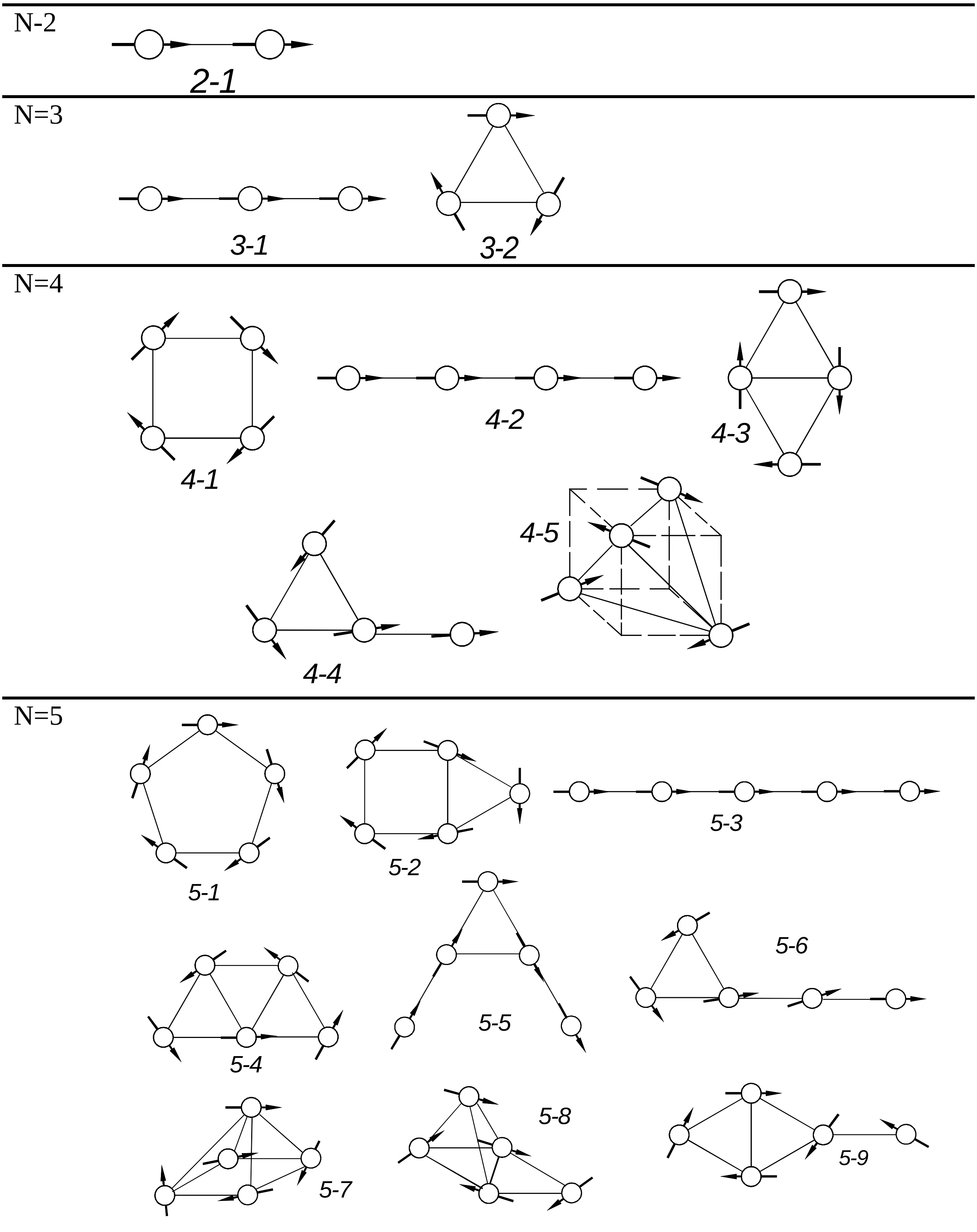
$$\frac{\Delta\varphi_M}{\Delta\varphi_D} \sim \frac{1}{4\pi}\frac{v}{c}\frac{u}{c}; \quad \frac{\Delta\varphi_T}{\Delta\varphi_M} \sim 4\pi\frac{N'R_c}{NL};$$

$$\frac{\Delta\varphi_G}{\Delta\varphi_D} \sim \frac{u^2N'R_c}{4\pi c^2 Nl}; \quad \frac{\Delta\varphi_G}{\Delta\varphi_T} \sim \frac{u}{4\pi v}.$$

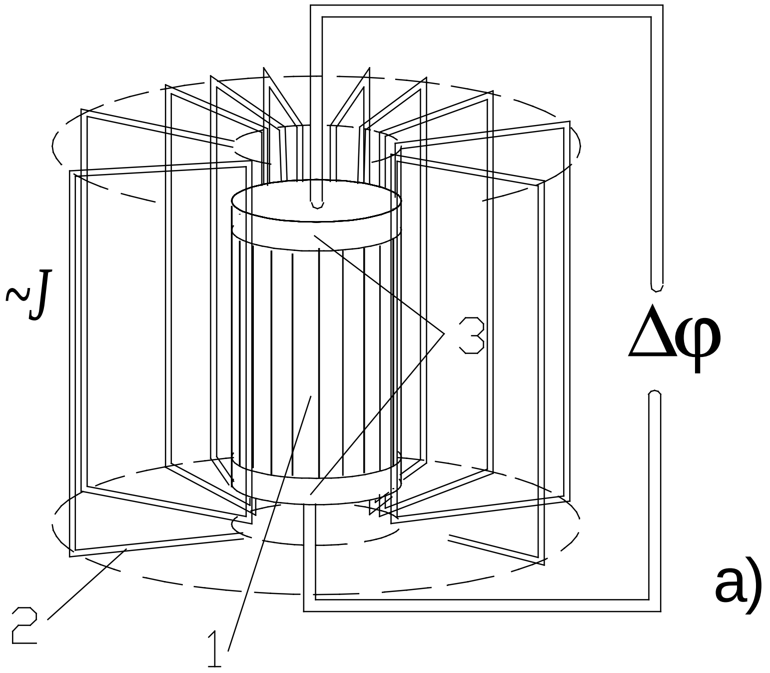
Applications of supertoroid electridynamics considered in details

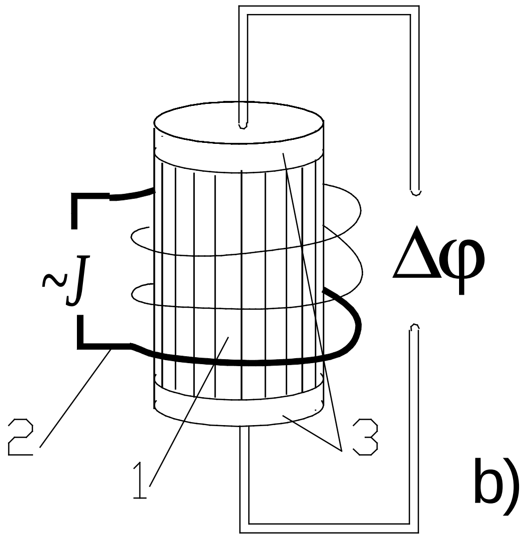
1. Magnetic toroid memory chip
2. Toroidness in antiferromagnetic theory
3. Origin of aromagnetism
4. Toroid susceptibility of aggregated magnetic particles suspension
5. Curie-Weiss behavior of aggregated magnetic fluid
6. Toroid relaxation in aggregated magnetic fluid
7. System of interacting spins dynamic in external alternating fields. Toroid echo. Toroid response. Toroid resonance absorption of energy
8. Structural theory of optical activity based on the toroid polarizability of chiral molecules

Minimum dipole-dipole-interaction energy N magnetic particles aggregates



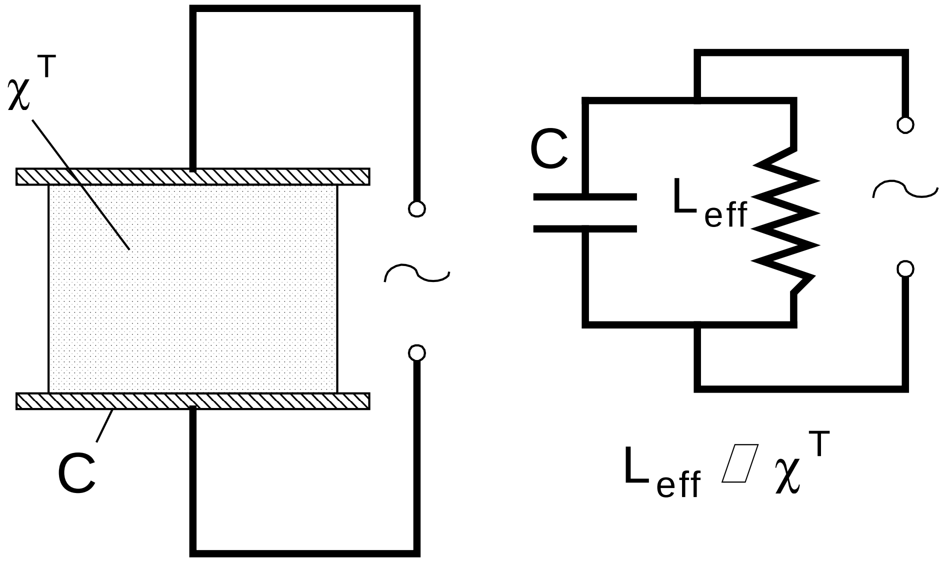
Schema for measuring toroid susceptibilities χ^{T} , χ^{TM}





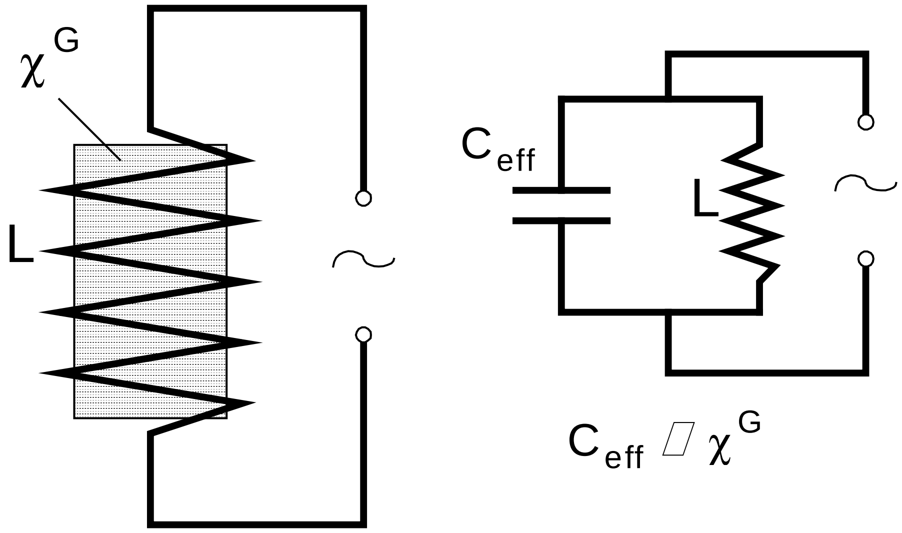
The cylinder-shaped sample 1 of substance that is having toroidal susceptibility χ^{T} (a) or cross susceptibility χ^{TM} (b) to be not equal to zero. Alternating current J in the coil induces vortex (a) or uniform (b) magnetic field on the sample. As a result toroid moment of the sample oscillates and potential $\Delta\varphi$ appears on the capacitor 3

Capacitor manifests low frequency inductance when filled by special material poses toroidal polarizability…



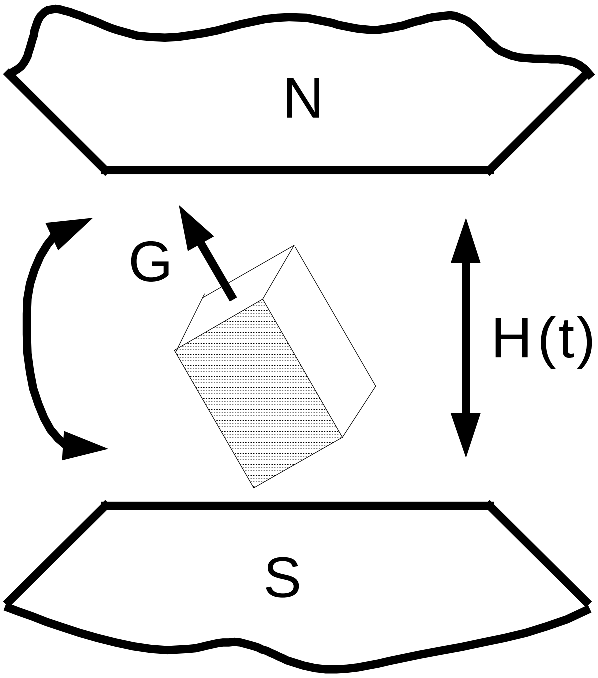
Capacitor C is filled by substance having toroidal susceptibility χ^T to be not equal to zero. Effective inductance L_{eff} on the equivalent circuit given on the right side is proportional to χ^T

…Inductance coil manifests low frequency capacity when filled by special material poses axial toroidal polarizability



Inductance coil L is filled by substance having toroidal susceptibility χ^G to be not equal to zero. Effective capacity C_{eff} on the equivalent circuit given on the right side is proportional to χ^G

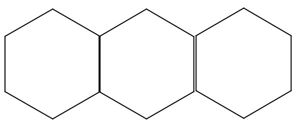
Aromagnetizm phenomena



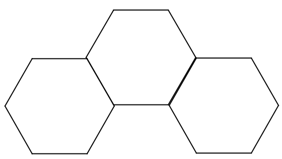
Reorientation of aromagnetic particle in altanative magnetic field H(t). It was found by experiment (N. Tolstoi & A.Spartakov) that particles have constant effective magnetic moment M_{eff} .

It was proposed (M.Martsenyuk & N. Martsenyuk) to explain the origin of aromagnetizm by axial toroid moment G that poses aromagnetic molecules

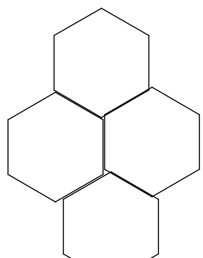
Molelecules of some aromagnetic substasnces



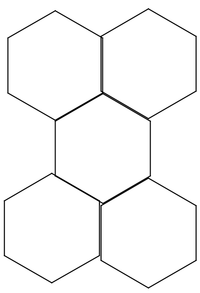
anthracen



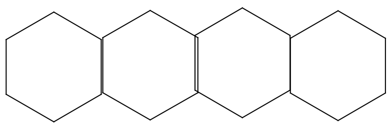
phenantren



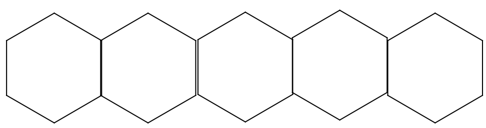
piren



perilen

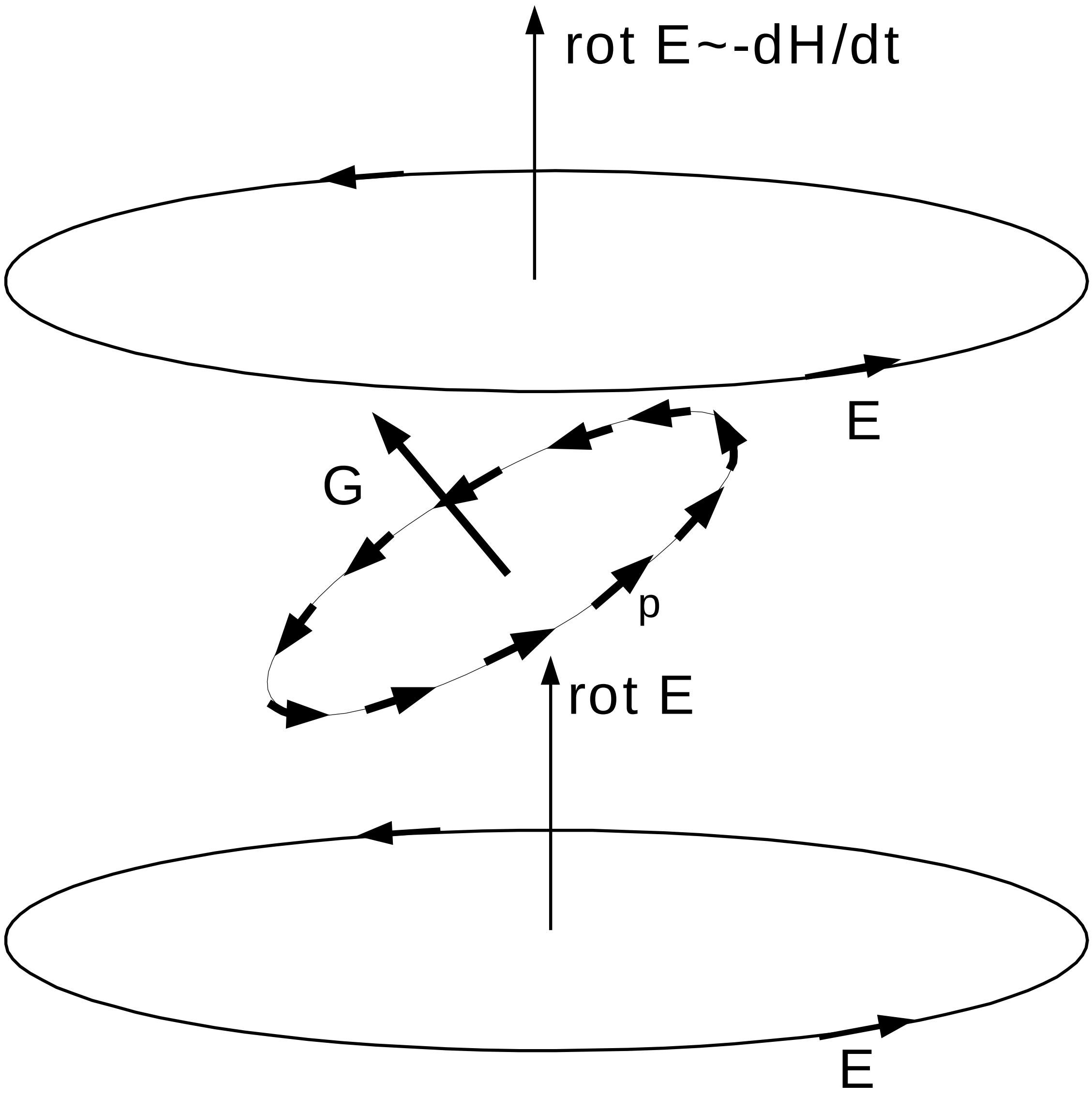


naphthacen



pentacen

Orientation of aromagnetic particle by alternating magnetic field



Moment of force on the particle is equal to

$$K = -\frac{1}{c}\left[G \times \frac{\partial H}{\partial t}\right]$$

Quantum chemistry calculation of dipole moments on atoms of aromagnetic molecule by method MOLCAO.

Dipole moments on a molecular fragment are depicted for the molecular orbital symmetry E_{2g} classified by irreducible representation of D_{6h} point group. In this quantum state molecule poses axial toroid moment as it is obviously follows from the picture

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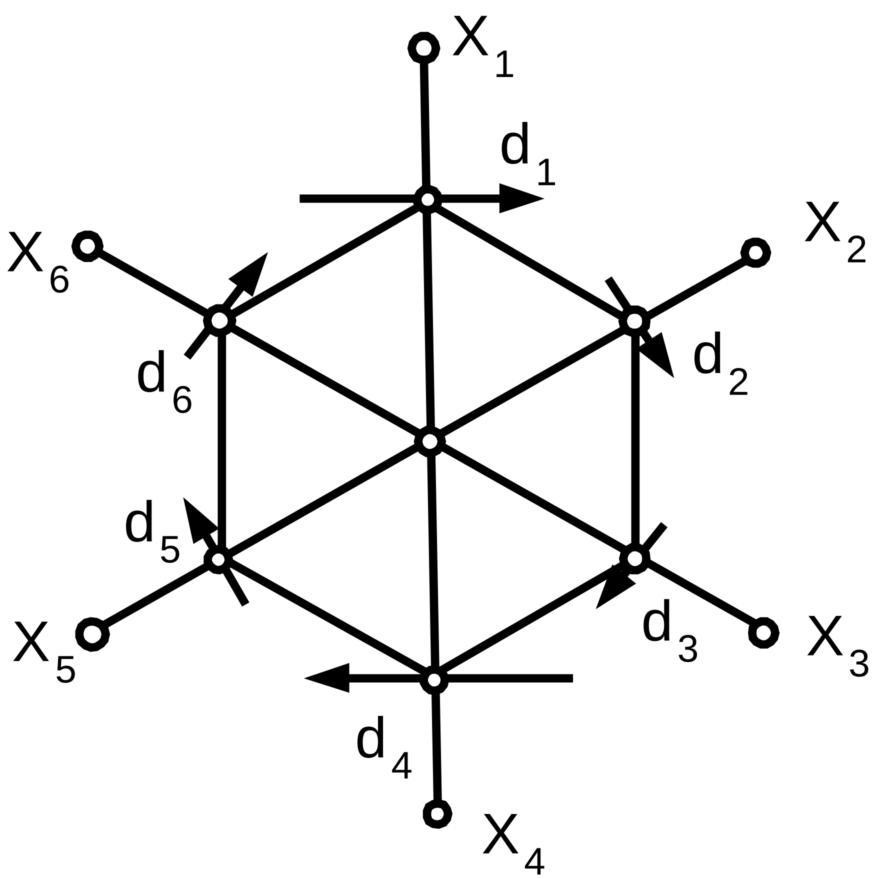
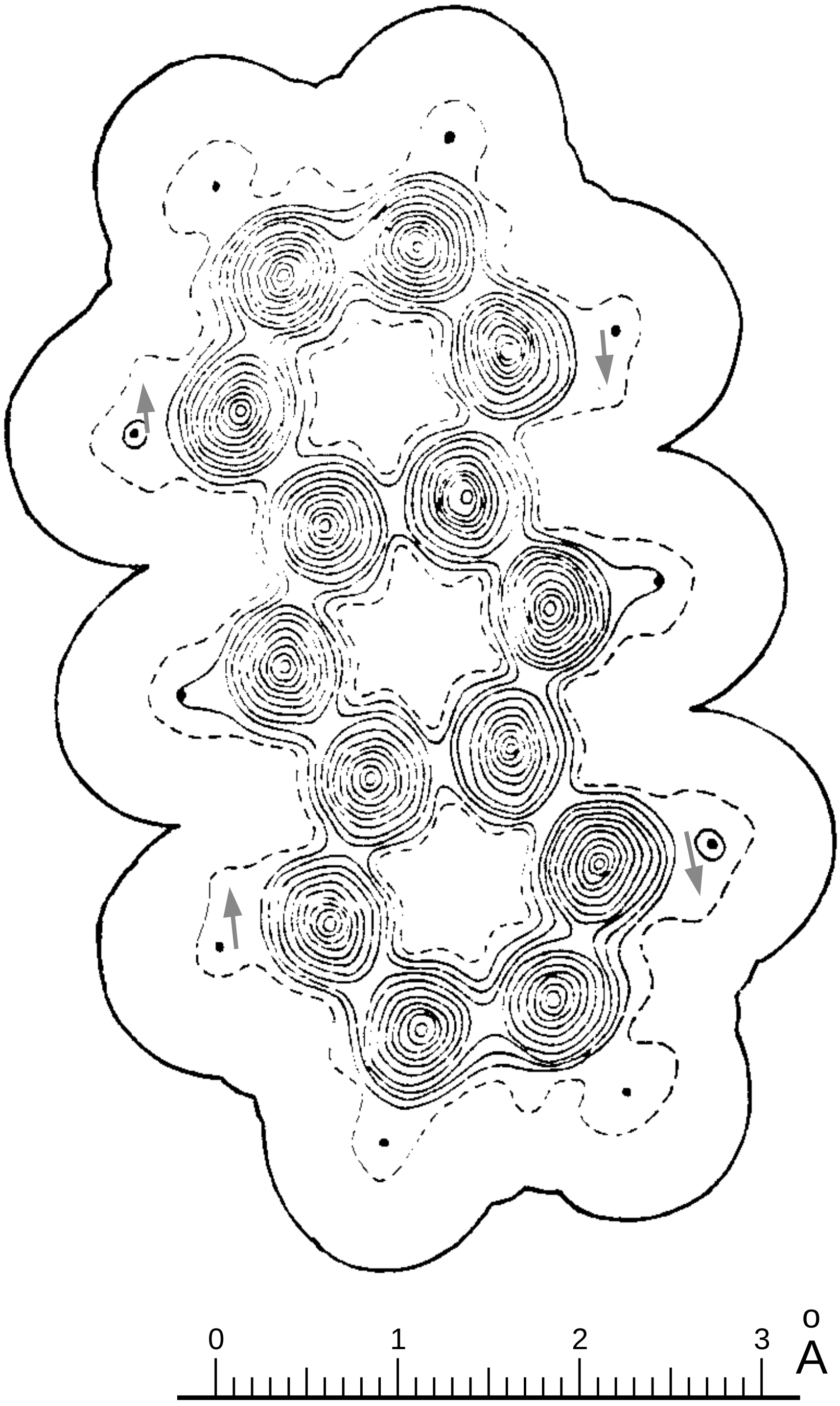
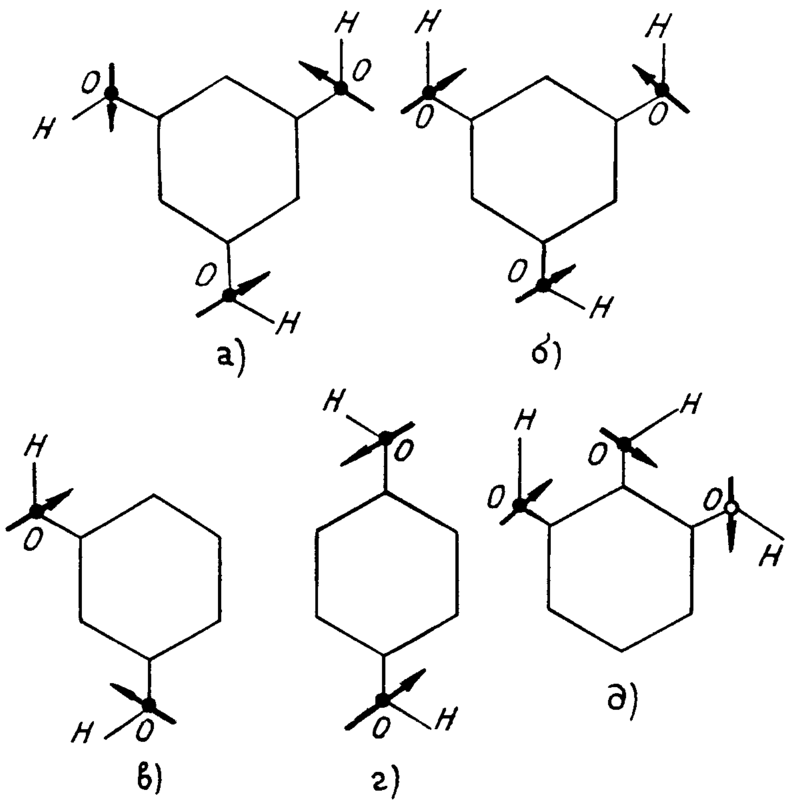


Electron density distribution in anthracene molecule $C_{14}H_{10}$ (X-ray experiment by Kitaigorodskiyi) Arrows visualize local dipole moments

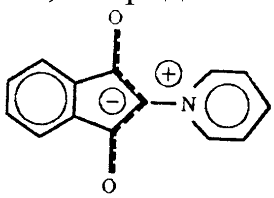


Possible aromagnetis: (1) molecular & (2) structural

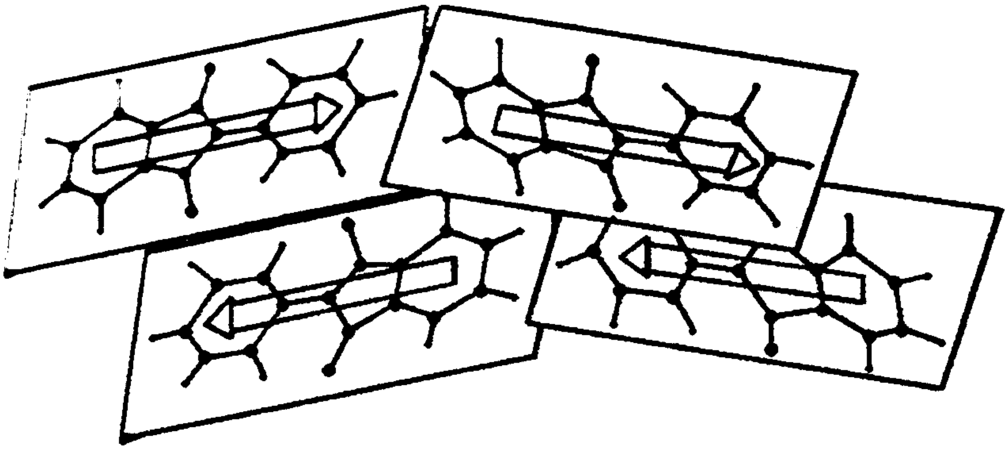
(1) а) флорглицин и его стереоизомер (б); в) ризорцин; г) гидрохинон; д) пирогаллол



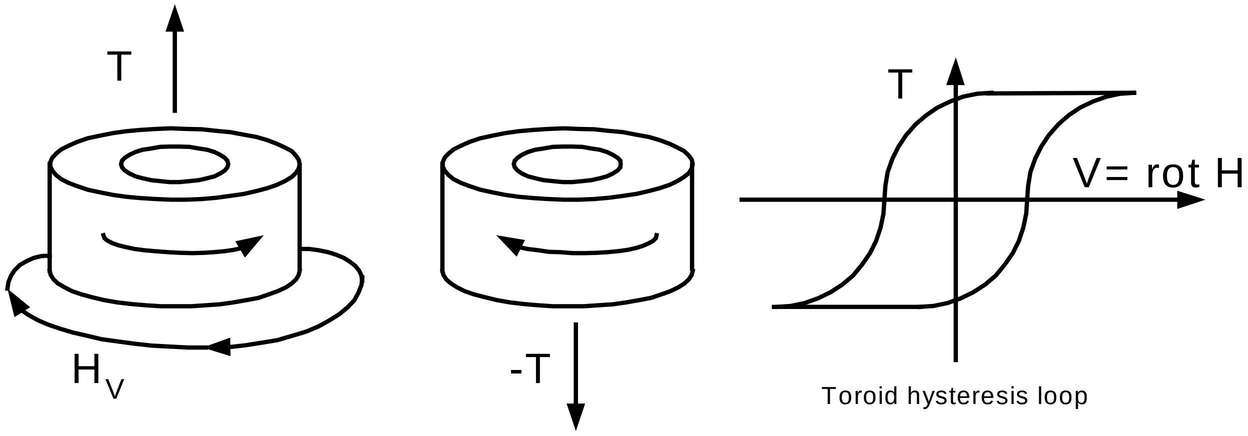
(2) Индандион-1,3 пиридиний бетаин (IPB)

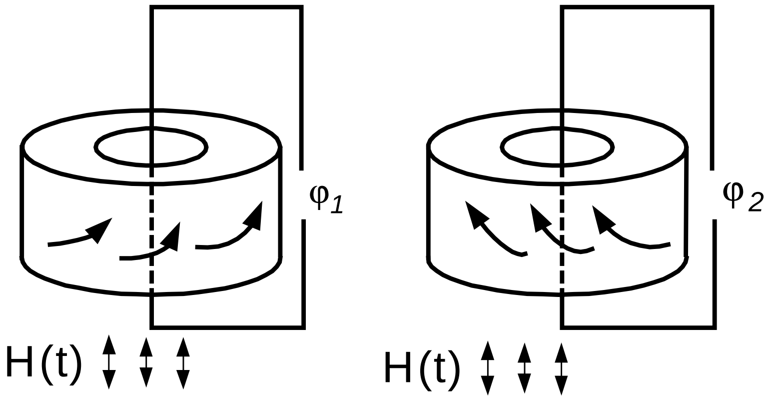


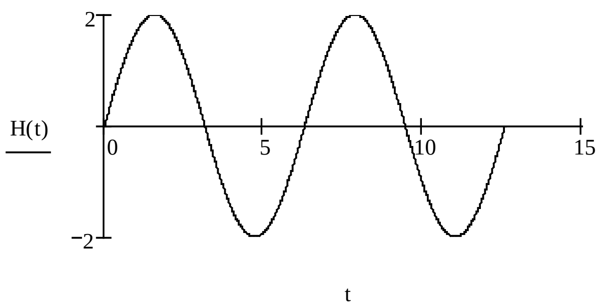
Structure of IPB crystal

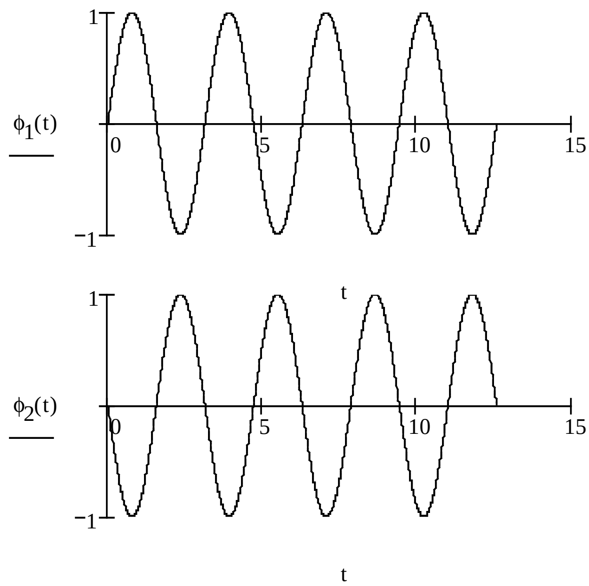


Toroid writing and reading principles

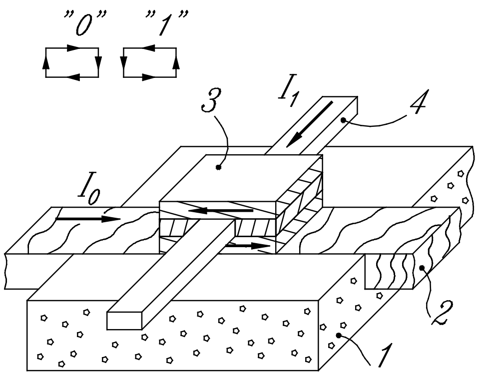




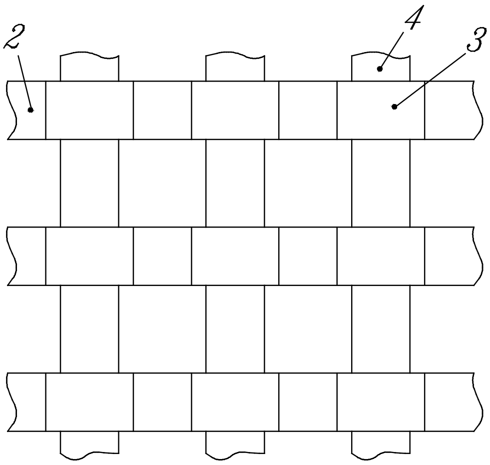




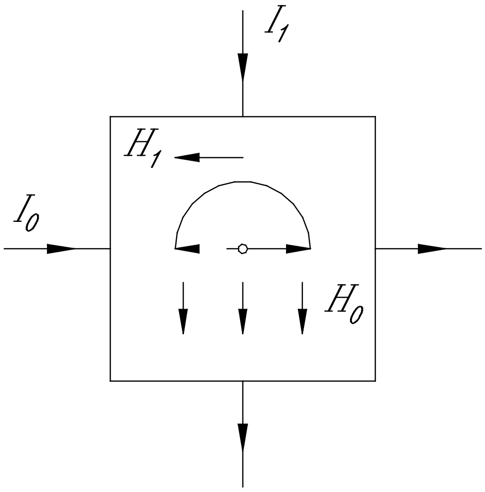
Magnetic toroid memory matrix



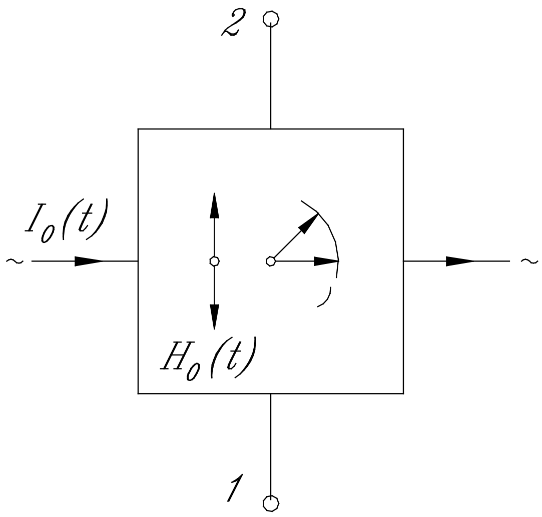
a) Toroid memory cell



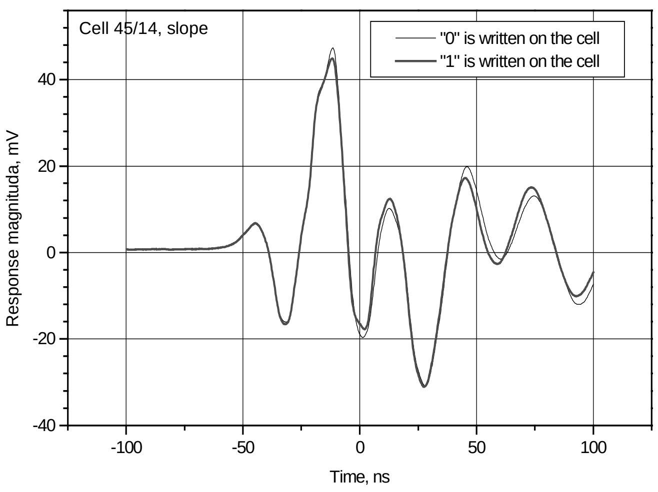
b) Toroid memory matrix



c) Writing on the cell

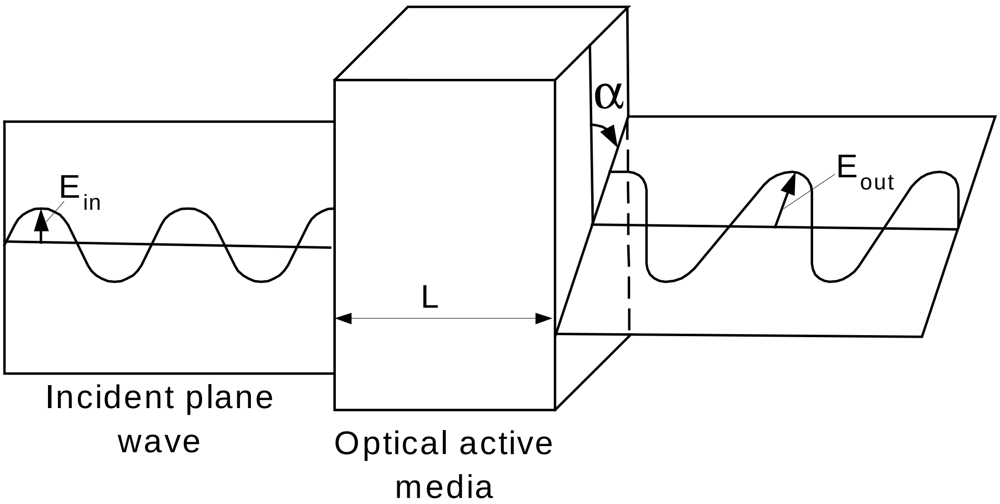


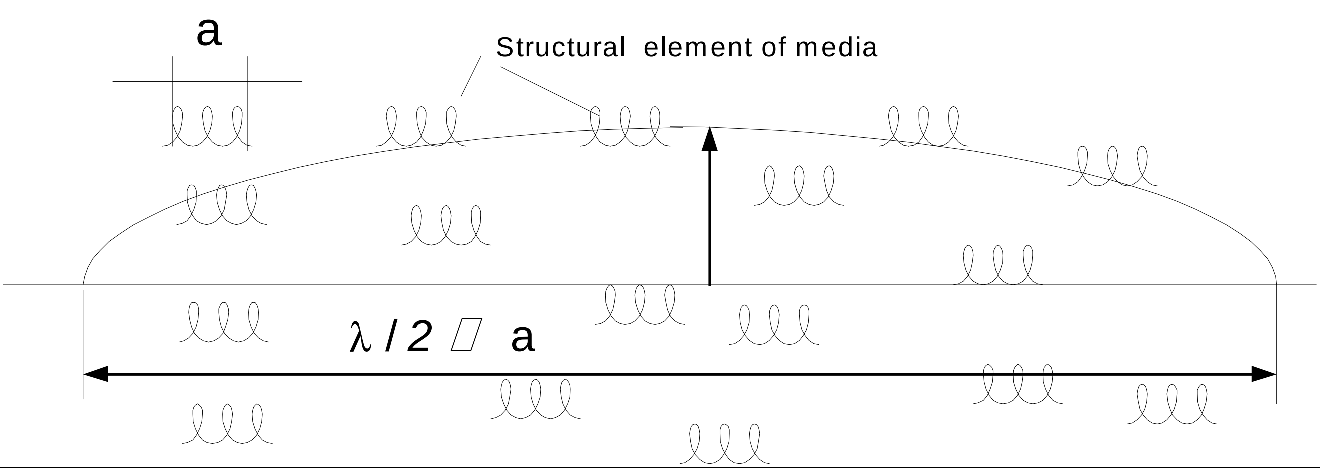
d) Reading from the cell



Response of 5×5 mcm memory cell on pulse of magnetic field created by pulse of current $I_0(t)$. The response depends on information that is written on the memory cell depicted on the fig a)

Toroid theory of optical activity of dielectric chiral media





The problem is to find how α depends on the properties and geometry of structural elements of media?

Existing theory:

1) Phenomenologycal $D_i = \varepsilon_{ik}^{(0)} E_k + \gamma_{ikl} \dfrac{\partial E_k}{\partial x_l}; \quad \gamma_{ikl} = \dfrac{c}{\omega} f e_{ikl}$

2) Quantum $\alpha \sim \langle \psi_1 | P | \psi_2 \rangle \langle \psi_1 | M | \psi_2 \rangle$
(other level of phenomenology)

Vector moments and cross polarizabilities γ and η

	$\vec{E}$	$rot\vec{E}$	$\vec{H}$	$rot\vec{H}$
$\vec{p}$	$\chi^{(p)}$	γ	$\alpha^{(pm)}$	$\alpha^{(p\tau)}$
$\vec{g}$	$\tilde{\gamma}$	$\alpha^{(g)}$	$\alpha^{(gm)}$	$\alpha^{(g\tau)}$
$\vec{m}$	$\alpha^{(mp)}$	$\alpha^{(mg)}$	$\alpha^{(m)}$	η
$\vec{\tau}$	$\alpha^{(\tau p)}$	$\alpha^{(\tau g)}$	$\tilde{\eta}$	$\alpha^{(\tau)}$

$$\vec{m}_{eff.} = \frac{1}{c}\dot{\vec{g}} + \vec{m};$$

$$\begin{cases} \vec{p} = \chi^{(p)} \cdot \vec{E} + \gamma \cdot rot\vec{E} \\ \vec{g} = \gamma' \cdot \vec{E} + \chi^{(g)} \cdot rot\vec{E} \end{cases}$$

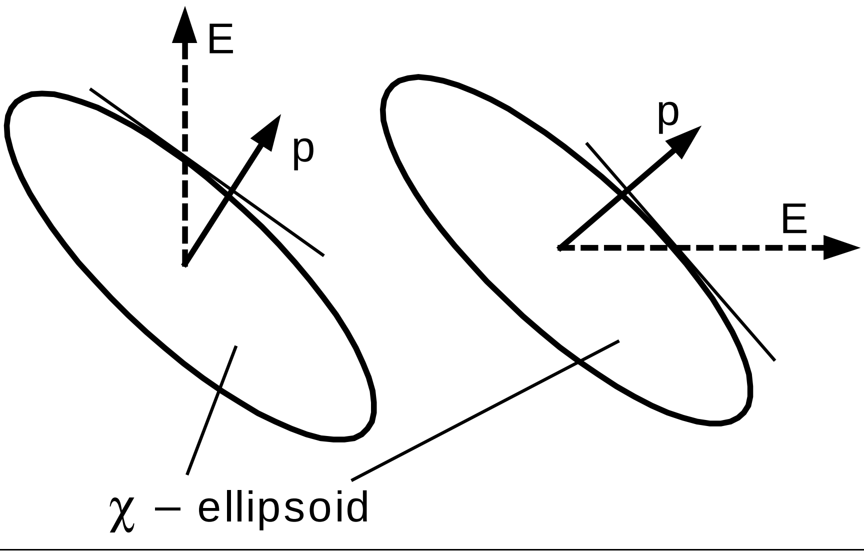
If γ **is a scalar:**

$$\alpha = -\frac{16\pi^3 Nl}{\lambda^2}\gamma$$

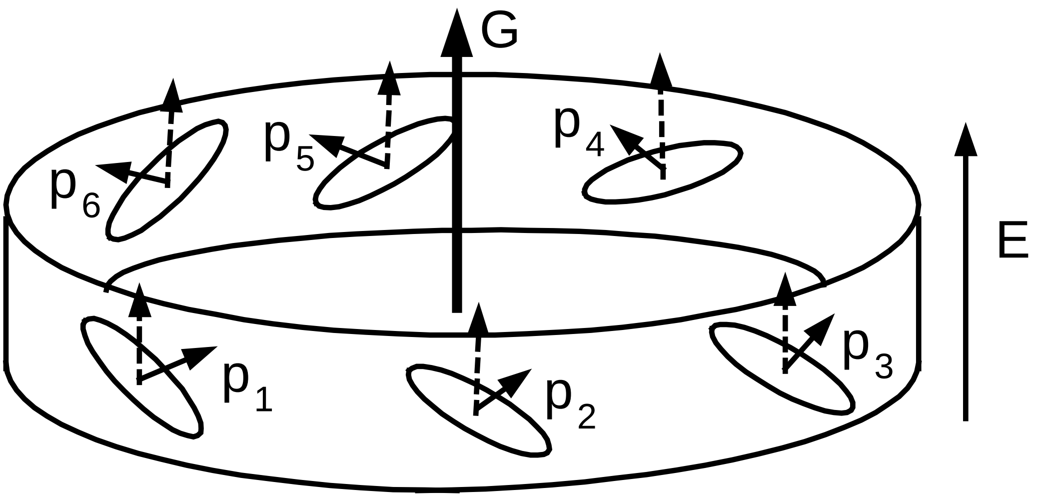
If γ is a **tensor**:

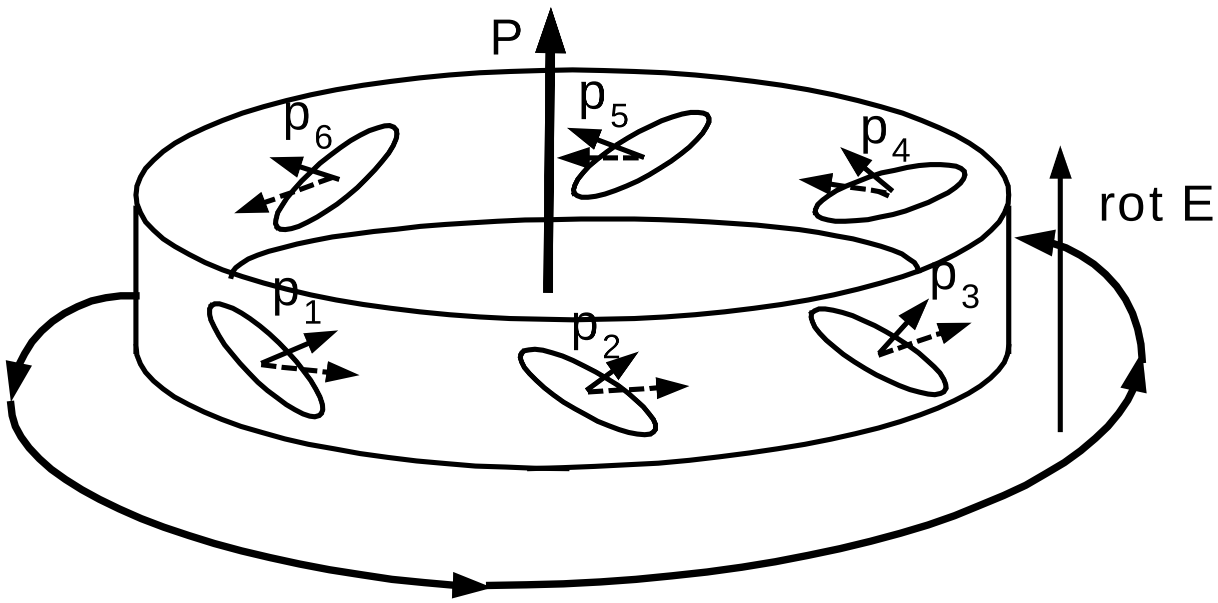
$$\alpha = -\frac{8\pi^3 Nl}{\lambda^2}(\gamma_{11} + \gamma_{22})$$

$$p_i = \chi_{ik}^{(p)} E_k$$



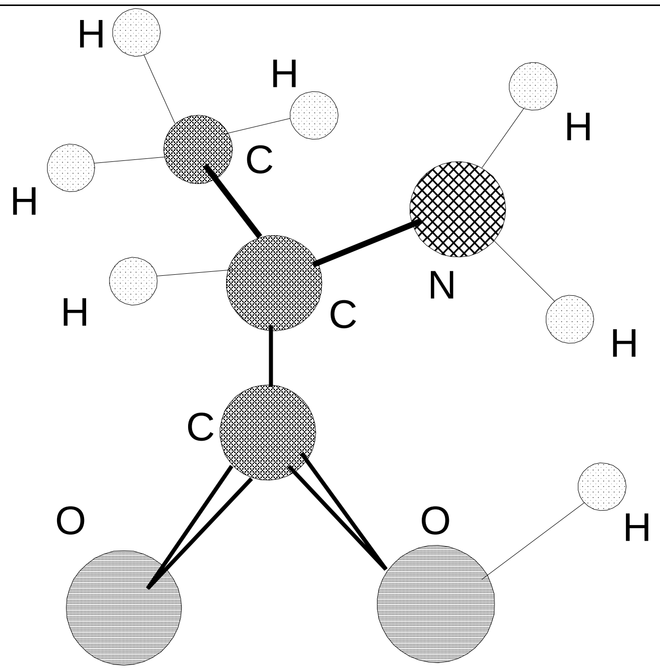
Qualitative explanation of optical activity effect using «cross» polarizabilities





Model of the body with the cross polarisability. Group of N dielectric ellipsoids have chiral relative deposition. In this case uniform elictric field E that is applied on the system induces the toroid moment G. And visa versa – nonuniform vortex field can induce dipole moment P.

Calculation of “cross” polarizability



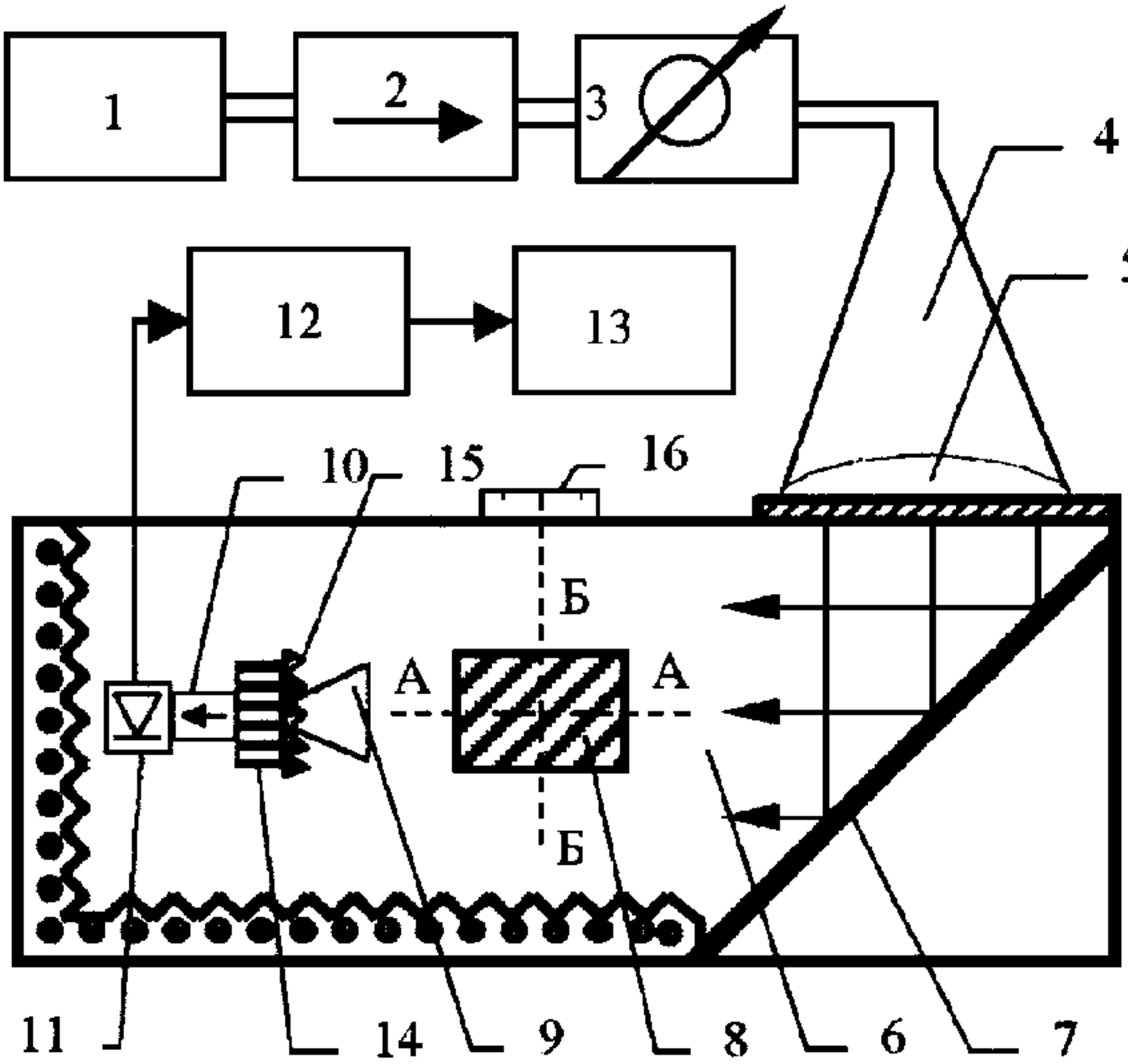
$CH_3CH(NH_2)COOH$ (Alanine)

Algorithm

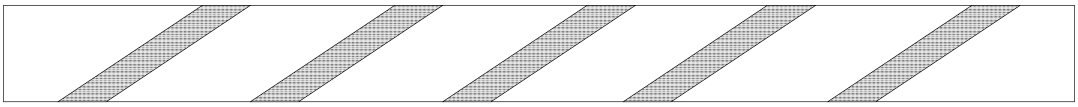
1. Find molecule geometry by on of the approximate methods (atom-atom/molecular dynamics or other)
2. Input atom polarizabilities estimated through atom refraction coefficients
3. Consider molecule in applied electric field as a system of interacting dipoles (Kirkwood method) and find atom dipole moments p_a in molecule
4. Calculate dipole P and toroid G moments for molecule as a whole and find cross and direct polarizabilities of a molecule

Экспериментальный удельный угол вращения плоскости поляризации				Теоретический удельный угол
Вещество	Растворитель	t (C^0)	$[\alpha]$ (град)	$[\alpha]$ (град)
D-Аланин	HCl	25	14.5	14.8
α-D-Глюкоза	H_2O	20	52.7	53
D-a-бромпропионовая кислота	без растворителя	20	29	22
D-a-бромфенилуксусная кислота	бензол	20	-145	-153

Microwave experiment for modeling molecular optical activity

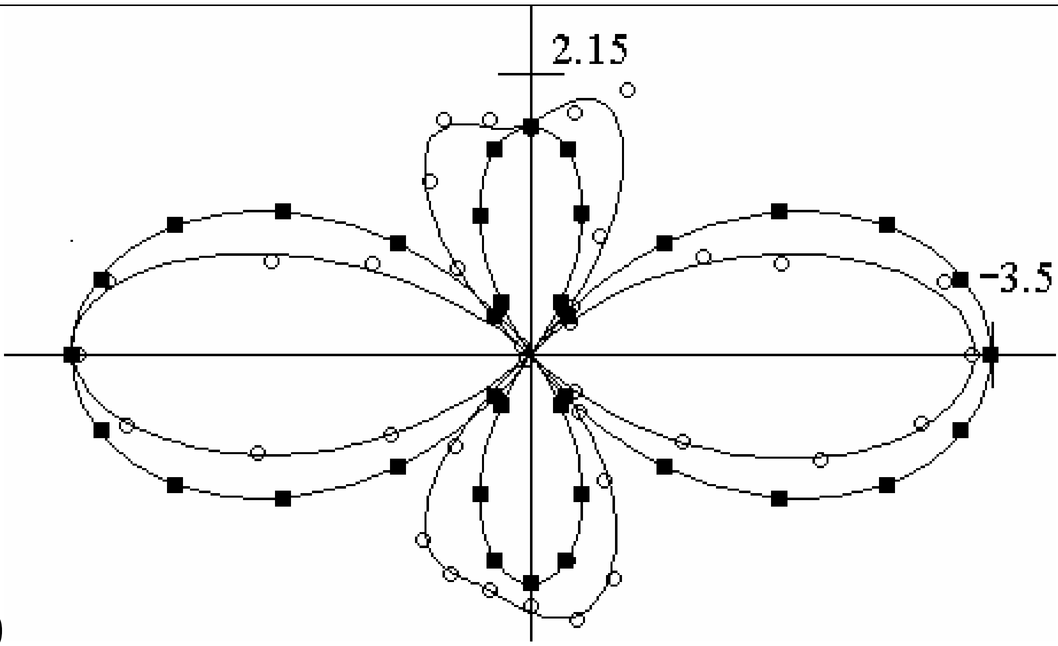


1-klystrone; 8 – sample: model of molecule made of artificial dielectric; 9- receive antenna; 11 – detector

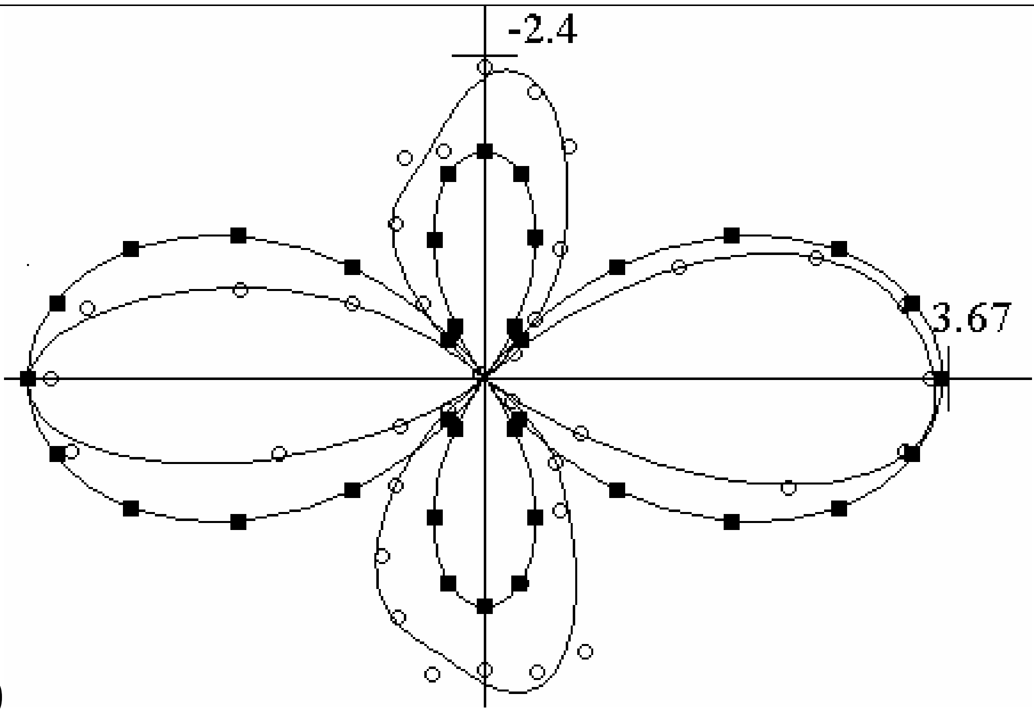


Model of structural element of chiral media. Foam plastic cylinder (d=1cm, L= 10 cm) bear spiral made of artificial dielectric which was prepared from mix of metal powder and paraffin.

Polar diagrams of dependence of rotary angle on the sample orientation (a) right spiral and (b) left spiral. Theory points a given by black squares. Spirals rotate 3 cm wave polarization in opposite directions



a)



b)

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From the Detection of the Magnetic Component of Optical Near-Fields to Nanotorus Plasmons Carrying a Magnetic Dipole Moment at Optical Frequencies

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Uni. Strasbourg	Prof. T. Ebbesen, Dr. E. Devaux
Uni. Toulouse	Prof. J. Weiner & team
NoE	Plasmo-Nano-Devices (16 partners)
STREP	ASPRINT (9 partners)

Running Supports

Regional Council of Burgundy

ACI : nanostructures et fonctionnalités

Equipe-Projet CNRS

GDR Optique atomique et nanostructures

European Commission

European Commission

Introduction : Why Torus Surface Plasmons ?

- Magnetic dipole moments at optical frequencies ??
- 1st indirect experimental evidence in NFO microscopy in relation with circular symmetry plasmon of nanostructured probe tip

=> Design of toroidal nanostructures to create magnetic dipole moments at optical frequencies ??

=> Consequence for quantum electrodynamics ??

Near-Field Optics

Scatterer's size a vs. incident wavelength λ

Macroscopic regime	$a >> \lambda$	ray optics, scalar approx.
Mesoscopic regime	$a \sim \lambda$	Maxwell equations (no approx.)
Microscopic regime	$a << \lambda$	non-retarded or dipolar approx.

C. Girard and A. Dereux, Rep. Prog. Phys. **59**, 657-699 (1996).

Near-Field Optics

Scatterer's size a vs. incident wavelength λ

Macroscopic regime	$a >> \lambda$	ray optics, scalar approx.
Mesoscopic regime	$a \sim \lambda$	Maxwell equations (no approx.)
Microscopic regime	$a << \lambda$	non-retarded or dipolar approx.

C. Girard and A. Dereux, Rep. Prog. Phys. **59**, 657-699 (1996).

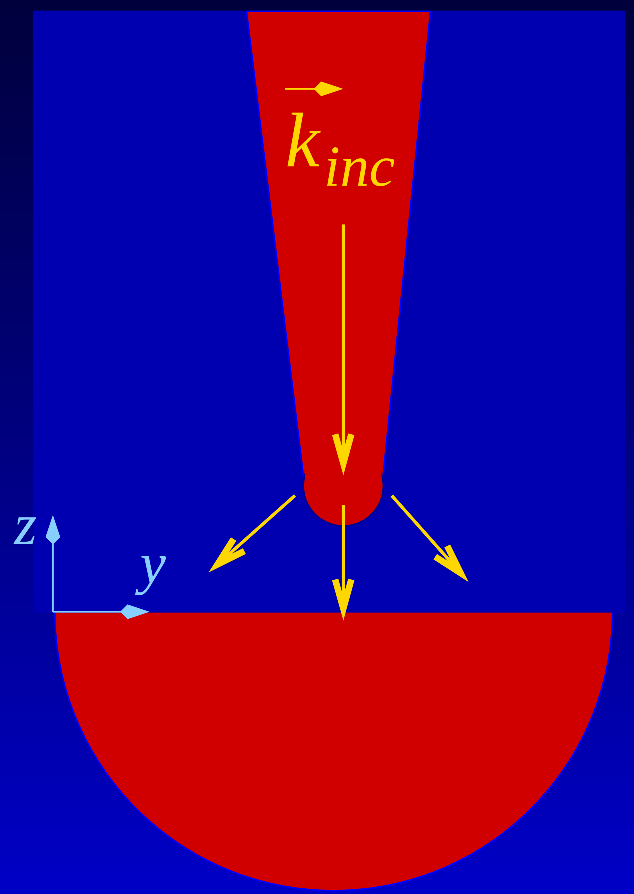
Near-Field Optics

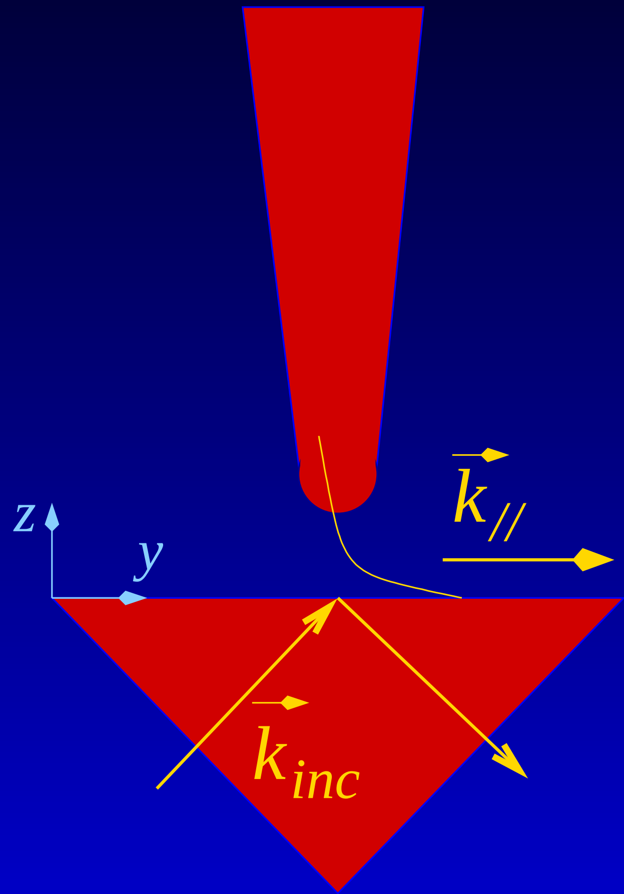
Scatterer's size a vs. incident wavelength λ

Macroscopic regime	$a >> \lambda$	ray optics, scalar approx.
Mesoscopic regime	$a \sim \lambda$	Maxwell equations (no approx.)
Nanoscopic regime	if λ visible	
Microscopic regime	$a << \lambda$	non-retarded or dipolar approx.

C. Girard and A. Dereux, Rep. Prog. Phys. **59**, 657-699 (1996).

Illumination vs collection mode

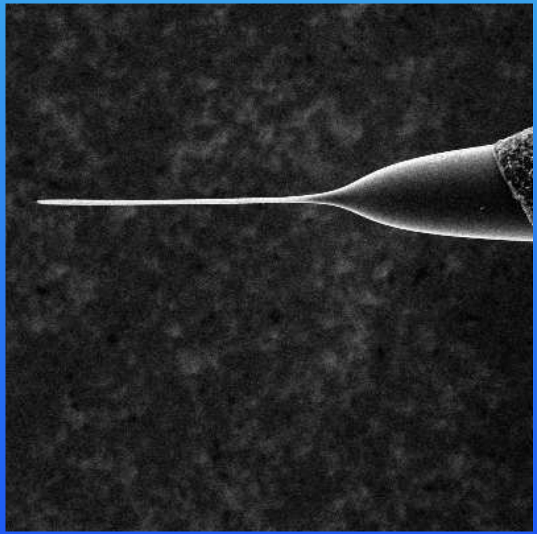


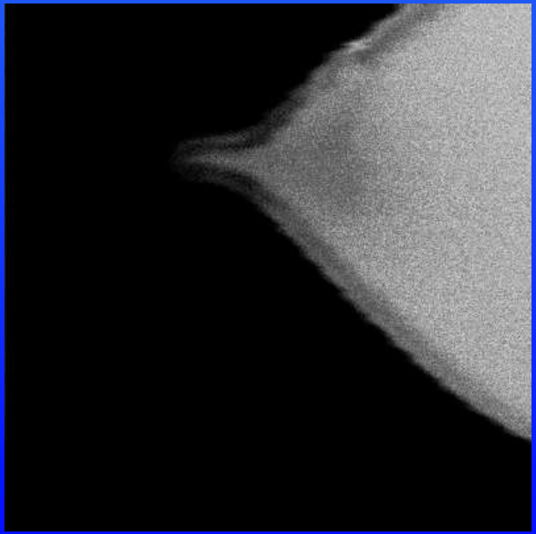


Tips fabrication

SEM images

300 μm x 300 μm

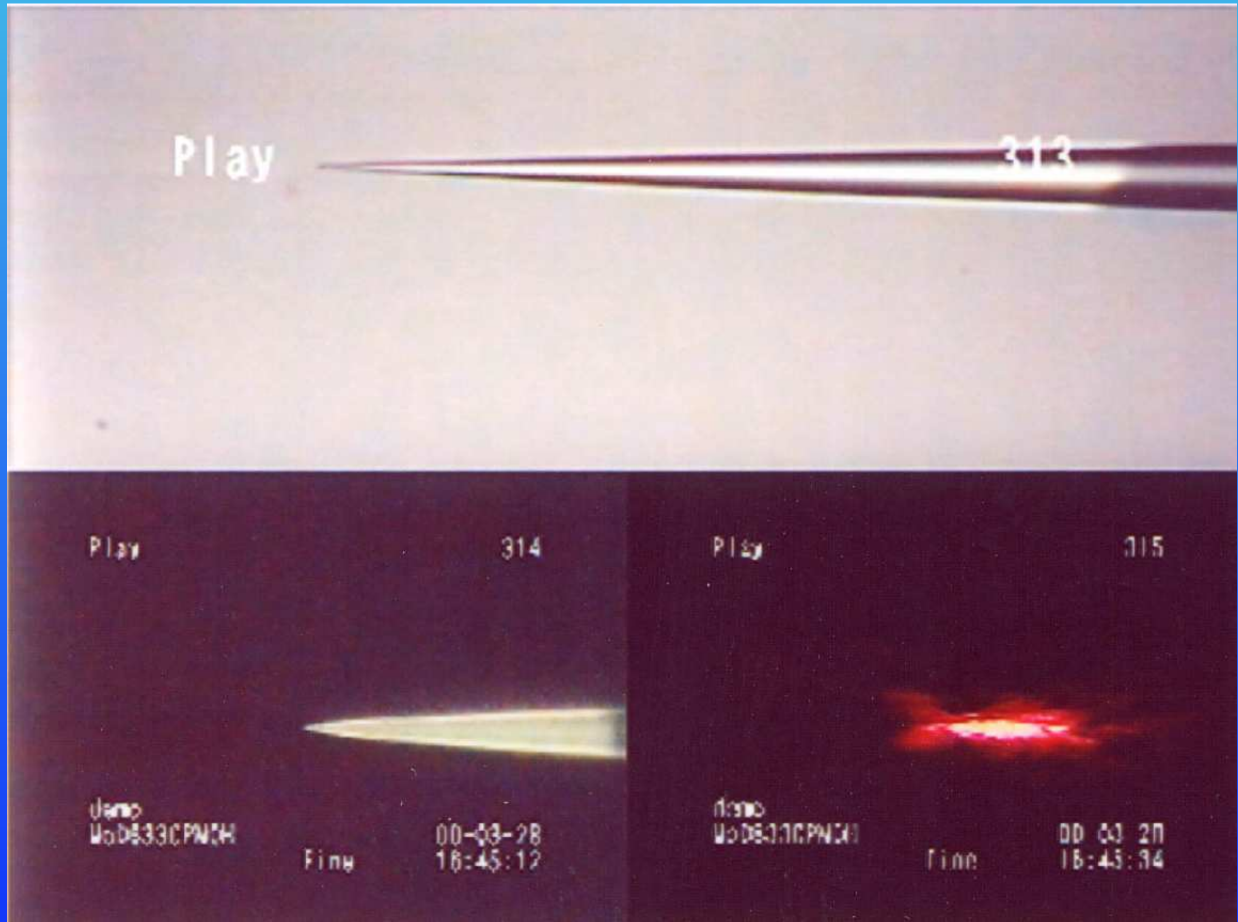




Tips fabrication : Y. Lacroute

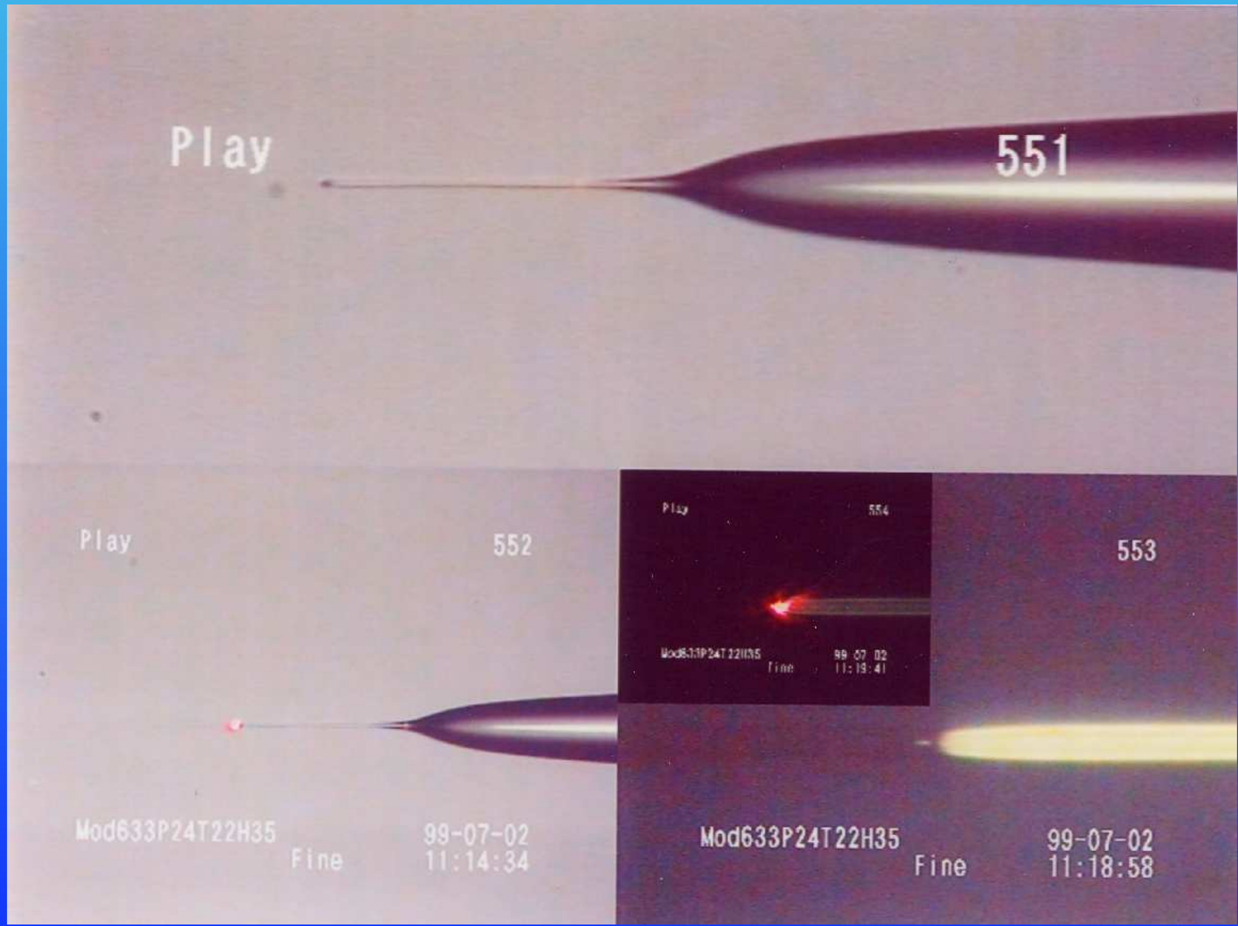
900 nm x 900 nm

Testing tips 1



- Y. Lacroute

Testing tips 2



- Y. Lacroute

How to define the sub-λ resolution ?

- Criterion : Recover the underlying shapes of material structures ?
- Problem : when NFO images look like the topography, they are probably artefactual (= disguised AFM images)

  B. Hecht *et al.*, J. Appl. Phys. **81**, 2492 (1997)

- Fact : Most NFO images do not look like the topography !
- Inverse scattering difficult
- => Another criterion ? Practical point of view

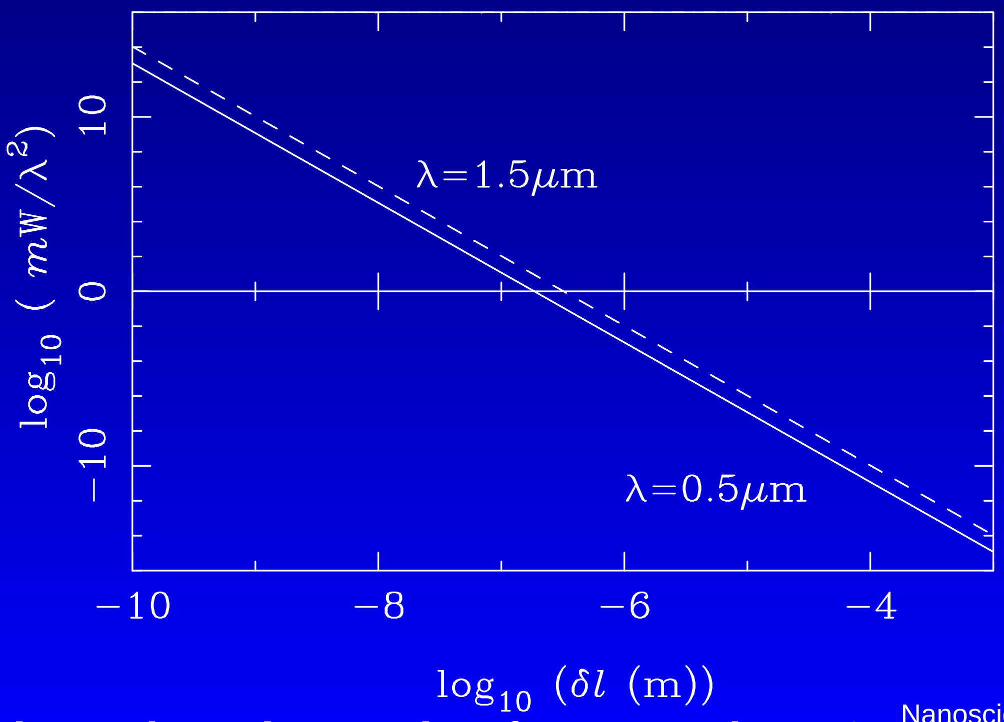
Sub λ detection & Heisenberg uncertainty

When the detection volume $(\delta l)^3$ is such that $\delta l << \lambda$,

$$\Delta x_i \; \Delta p_i \;\; \geq \;\; \hbar$$

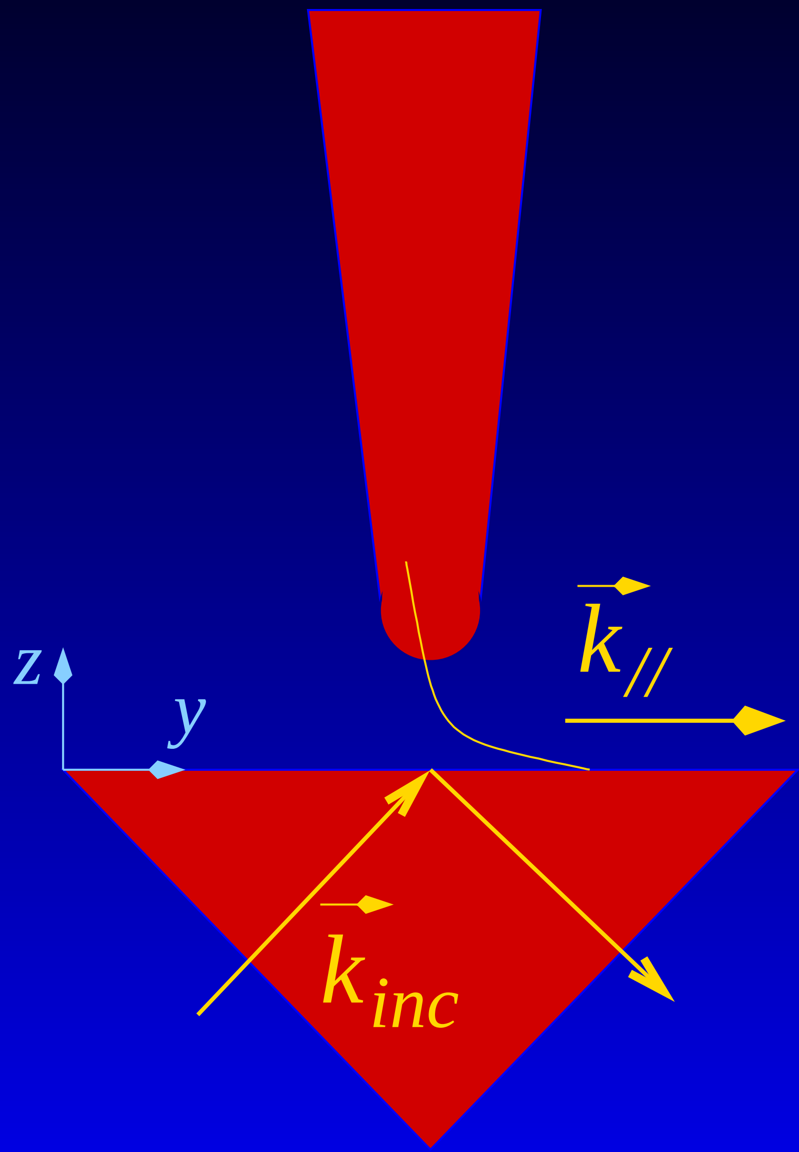
leads to (with cyclic permutation of $(i, j = x, y, z)$) :

$$\Delta E_i \; \Delta H_j \;\; \geq \;\; \frac{\hbar}{2} \, \frac{c^{\,2}}{(\delta l)^4}$$

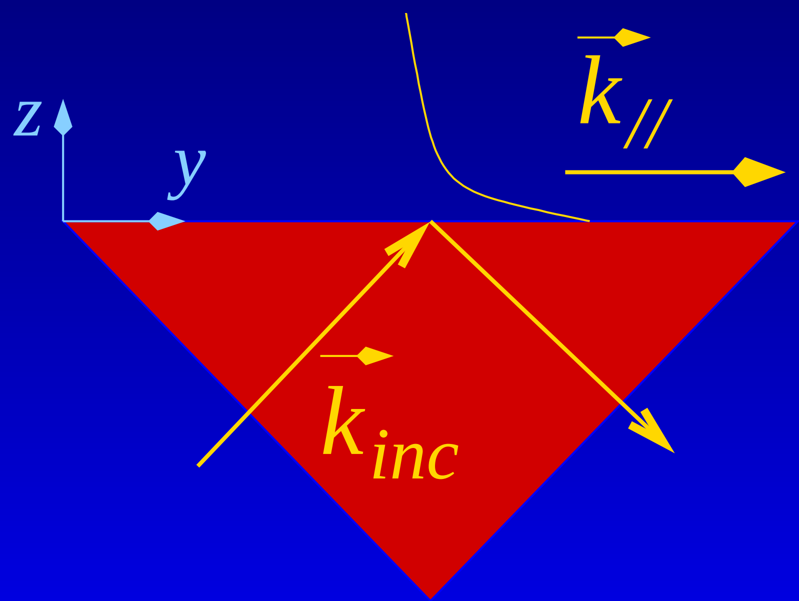


- W. Heisenberg, *Physical Principles of Quantum Theory* (1930).

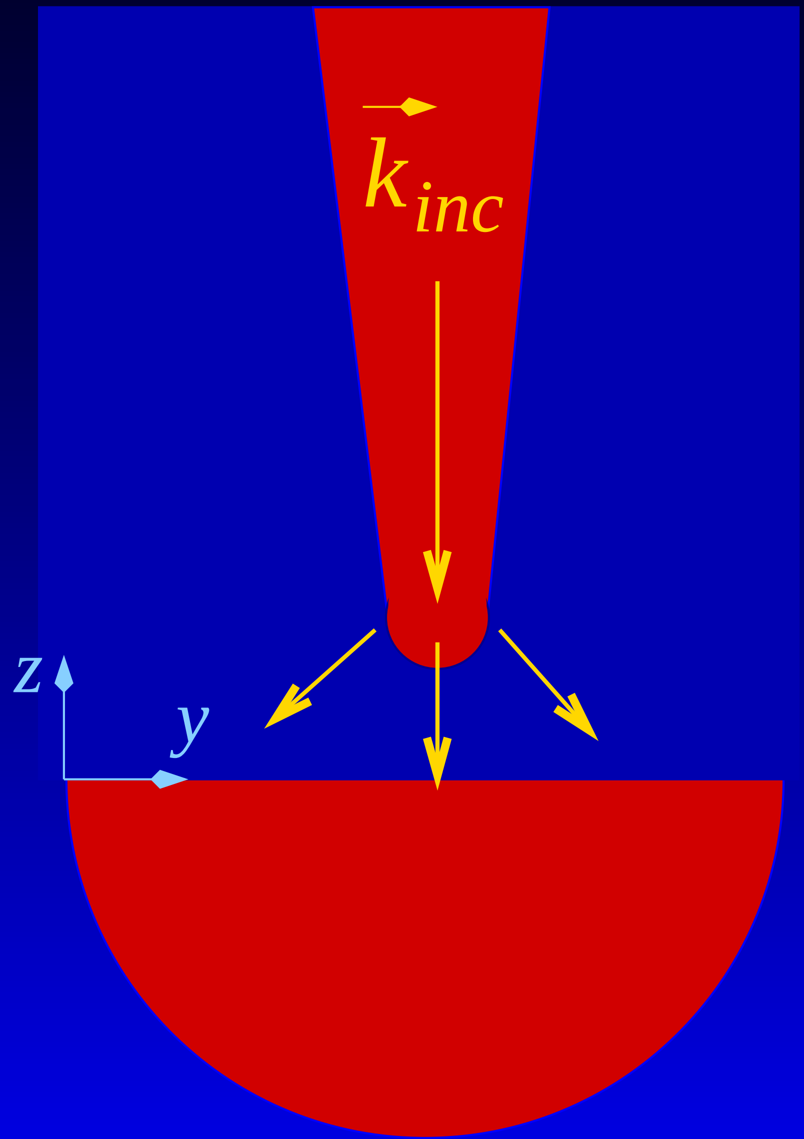
Interpretation of PSTM images



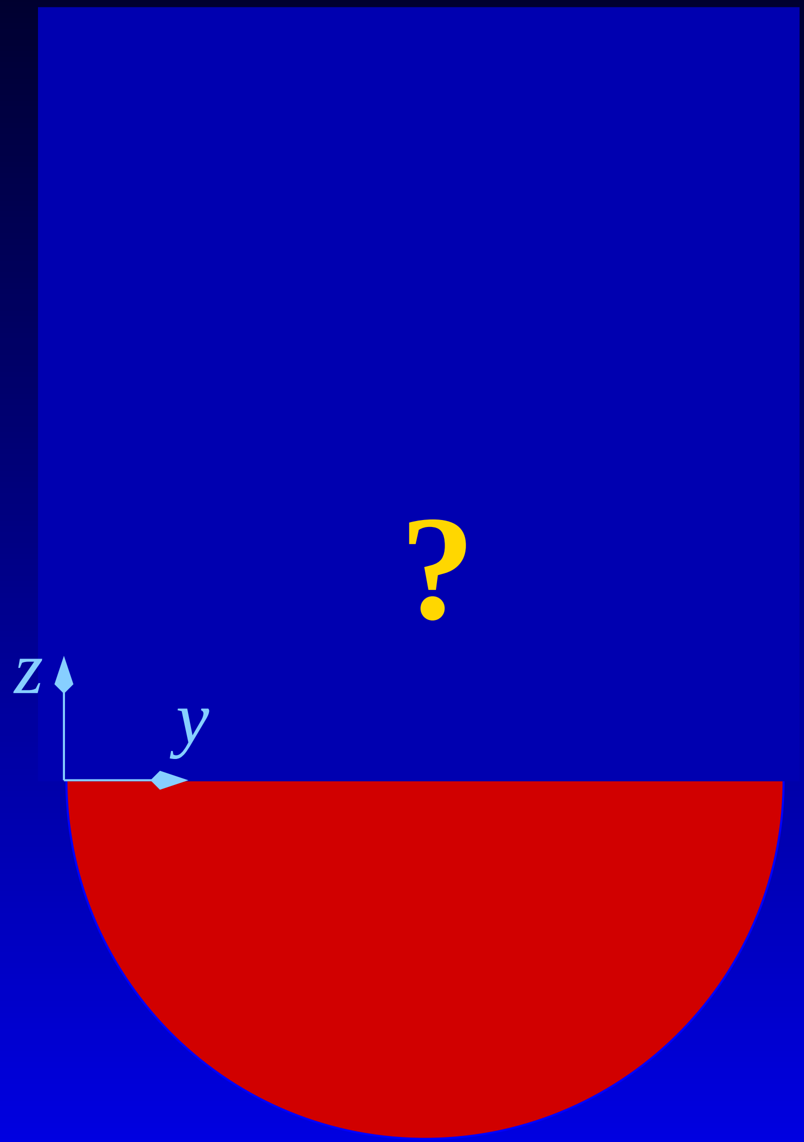
Interpretation of PSTM images



Interpretation of SNOM images



Interpretation of SNOM images



Direct interpretation of the images ?

- Are the images recorded by PSTM in agreement with the distributions of the electric and/or magnetic near–fields scattered by the sample surfaces, as computed without including any tip?

Direct interpretation of the images ?

- Are the images recorded by PSTM in agreement with the distributions of the electric and/or magnetic near–fields scattered by the sample surfaces, as computed without including any tip?
- Is there any link between the images obtained by SNOM and the ω-resolved distribution of the electromagnetic local density of states $\rho(\vec{r}, \omega)$ close to the sample surface, as computed without including any tip?

A. Dereux, C. Girard, J.C. Weeber, J. Chem. Phys. **112**, 7775 (2000)

Scattering theory

With the usual $\exp(\;\; i\omega t)$ time (t) dependence, $\vec{r}$ being a vector in direct space and ω being the angular frequency, the vector wave equation issued from Maxwell's equations (SI units: $c = \sqrt{\epsilon_0 \; \mu_0}$ is the speed of light in vacuum):

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}(\vec{r}) + \frac{\omega^2}{c^2} \overleftrightarrow{\epsilon}(\vec{r}) \; \vec{E}(\vec{r}) = 0$$

may be cast as

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}(\vec{r}) + q^2 \; \vec{E}(\vec{r}) = \overleftrightarrow{V}(\vec{r}) \; \vec{E}(\vec{r})$$

with

$$q^2 = \frac{\omega^2}{c^2} \; \epsilon_{ref} \text{ and } \overleftrightarrow{V}(\vec{r}) = \frac{\omega^2}{c^2} \; \left(\overleftrightarrow{1} \; \epsilon_{ref} \quad \overleftrightarrow{\epsilon}(\vec{r}) \right)$$

Lippmann-Schwinger equation

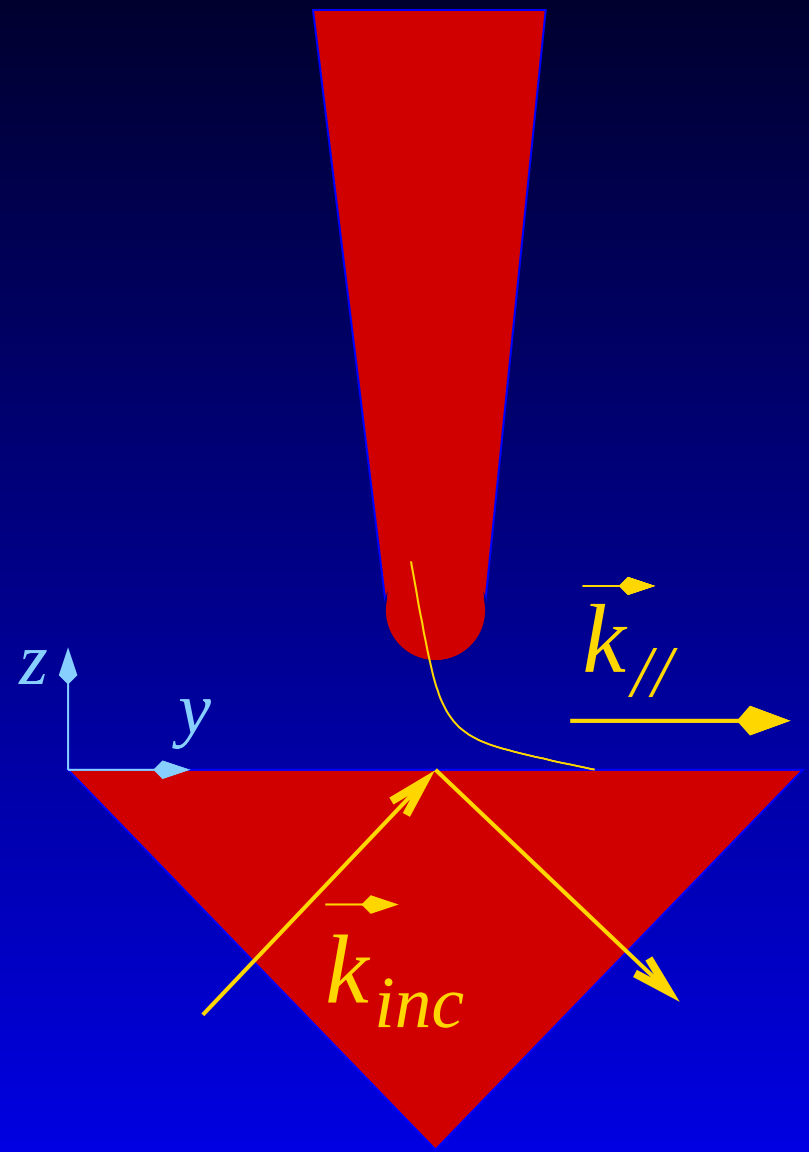
$$\vec{E}(\vec{r}) = \vec{E}_0(\vec{r}) + \int_V d\vec{r}\,'\ \overleftrightarrow{G}_0(\vec{r},\vec{r}\,')\ \overleftrightarrow{V}(\vec{r}\,')\ \vec{E}(\vec{r}\,')$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}_0(\vec{r}) + q^2\ \vec{E}_0(\vec{r}) = 0$$

$$\vec{\nabla} \times \vec{\nabla} \times \overleftrightarrow{G}_0(\vec{r},\vec{r}\,') + q^2\ \overleftrightarrow{G}_0(\vec{r},\vec{r}\,') = \overleftrightarrow{1}\ \delta(\vec{r} \quad \vec{r}\,')$$

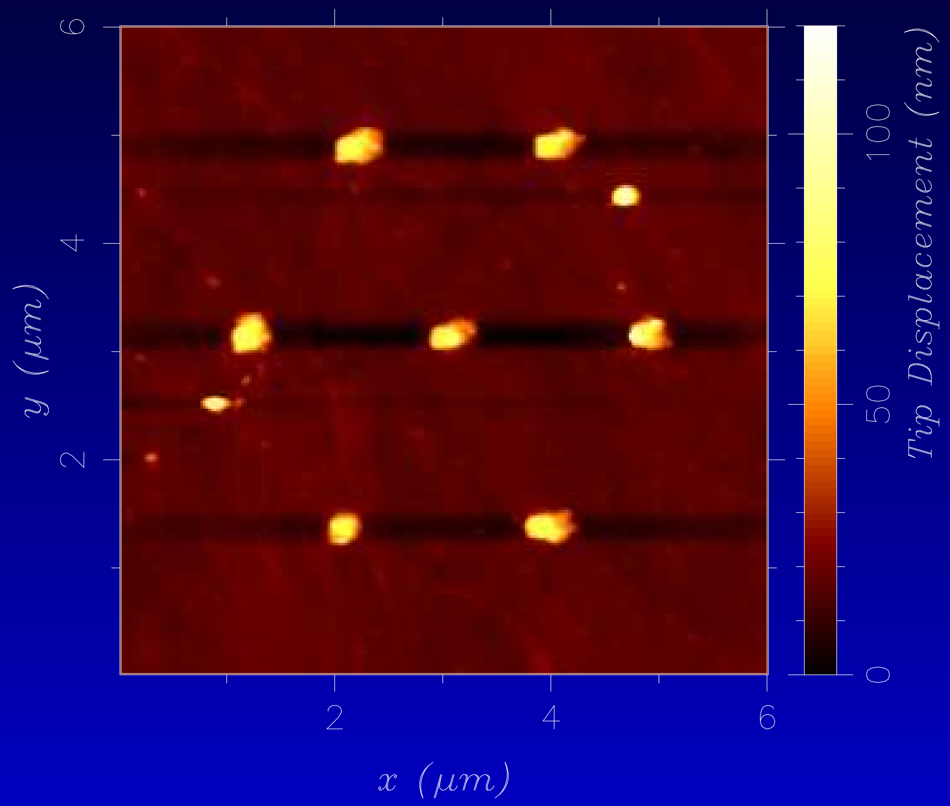
$$\vec{H}(\vec{r}) = \vec{H}_0(\vec{r}) + \frac{1}{\mu_0 c}\int_V d\vec{r}\,'\ \overleftrightarrow{Q}_0(\vec{r},\vec{r}\,')\overleftrightarrow{V}(\vec{r}\,')\vec{E}(\vec{r}\,')$$

Experimental tests : PSTM images

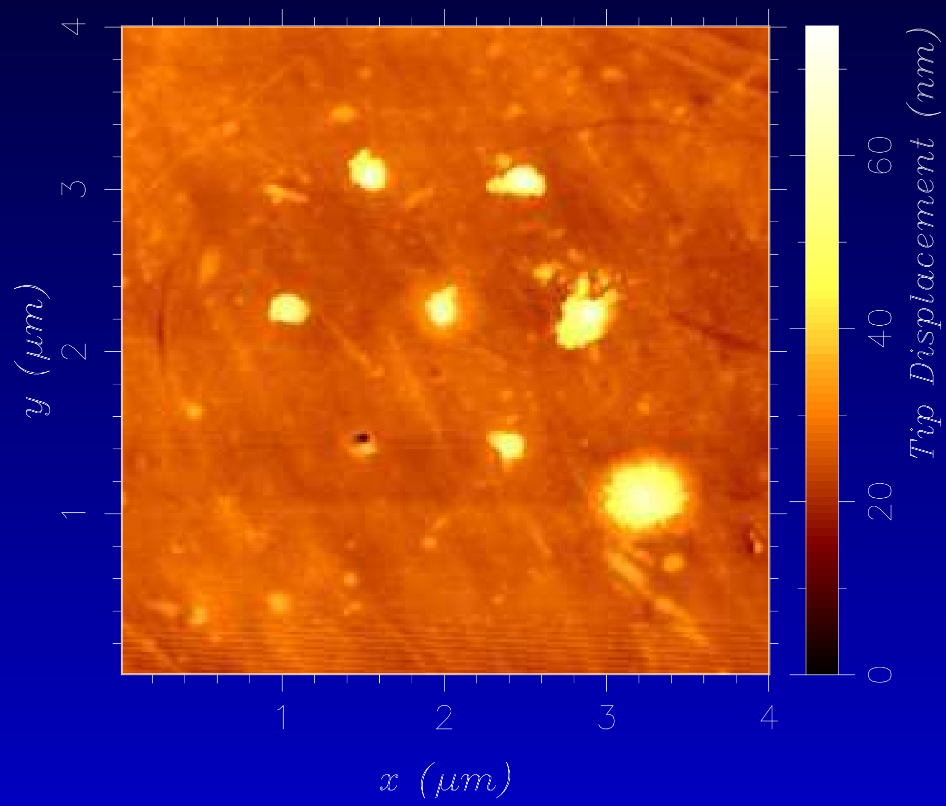


From Mesoscopic to Nanoscopic regime

Glass pads 250 nm × 250 nm × 100 nm

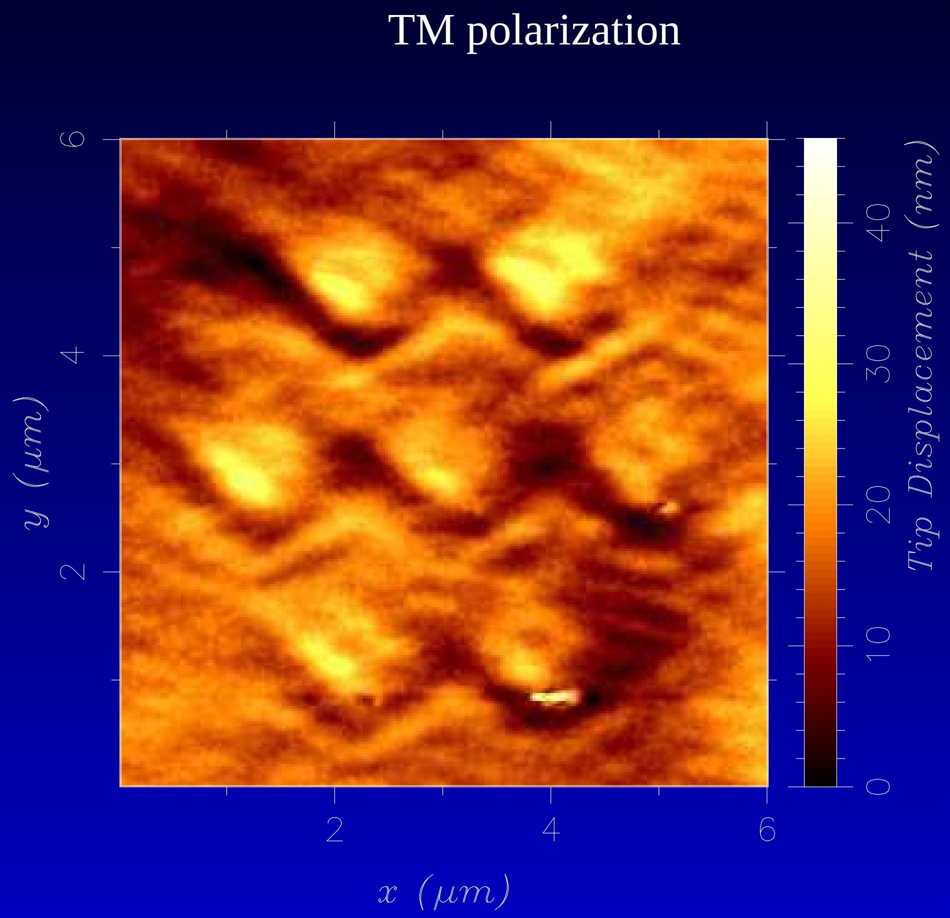


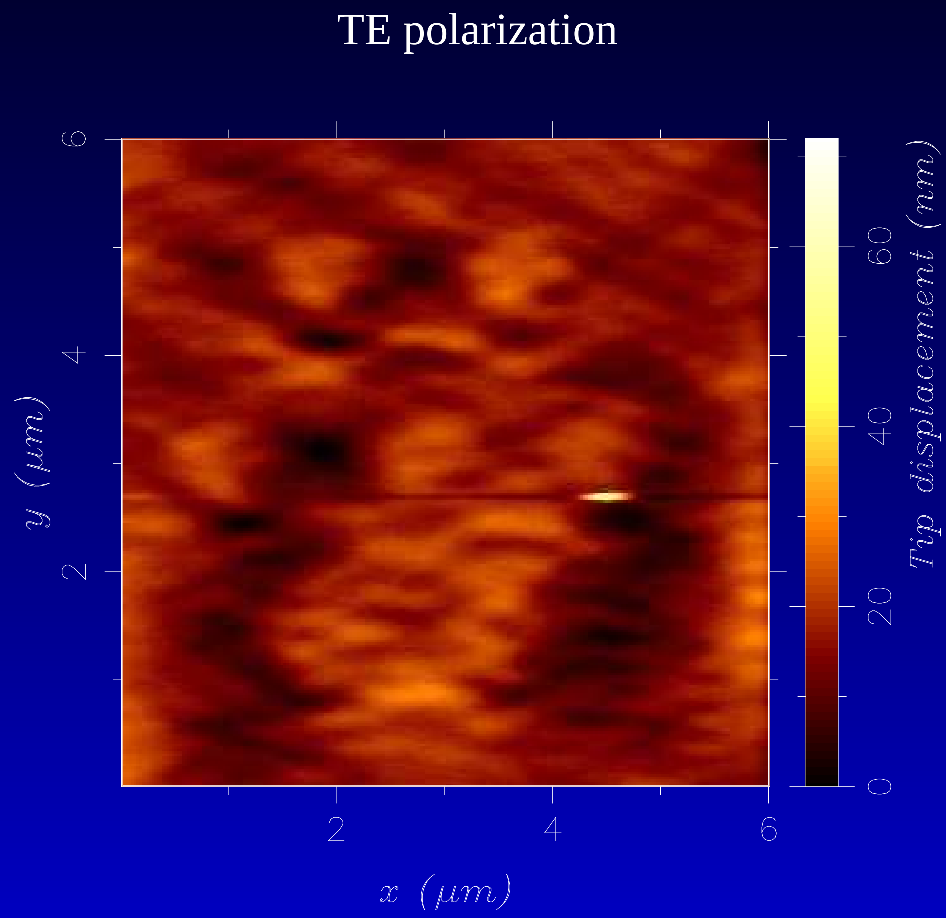
Glass pads 100 nm × 100 nm × 70 nm



- Samples on glass substrates by Y. Chen (L2M - Bagneux)
- J. C. Weeber, E. Bourillot, A. Dereux *et al.*, Phys. Rev. Lett. **77**, 5332 (1996)

Mesoscopic dielectric sample





- Dielectric tip ; $\lambda = 633$ nm ; $\theta = 60$ deg
- J. C. Weeber, E. Bourillot, A. Dereux *et al.*, Phys. Rev. Lett. **77**, 5332 (1996)

Nanoscopic dielectric sample

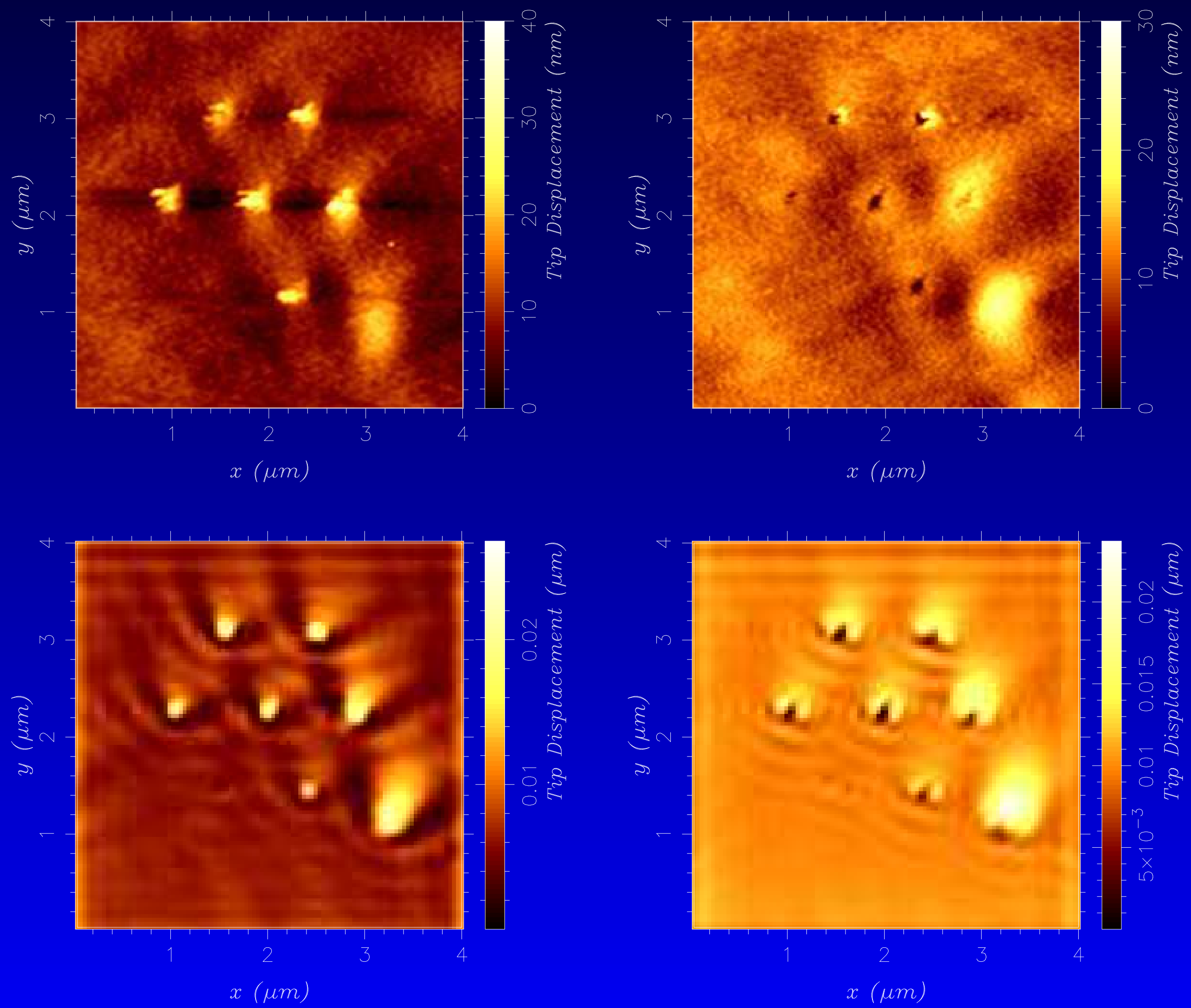
DIELECTRIC TIP

TM polarization

TE polarization

Isointensity PSTM

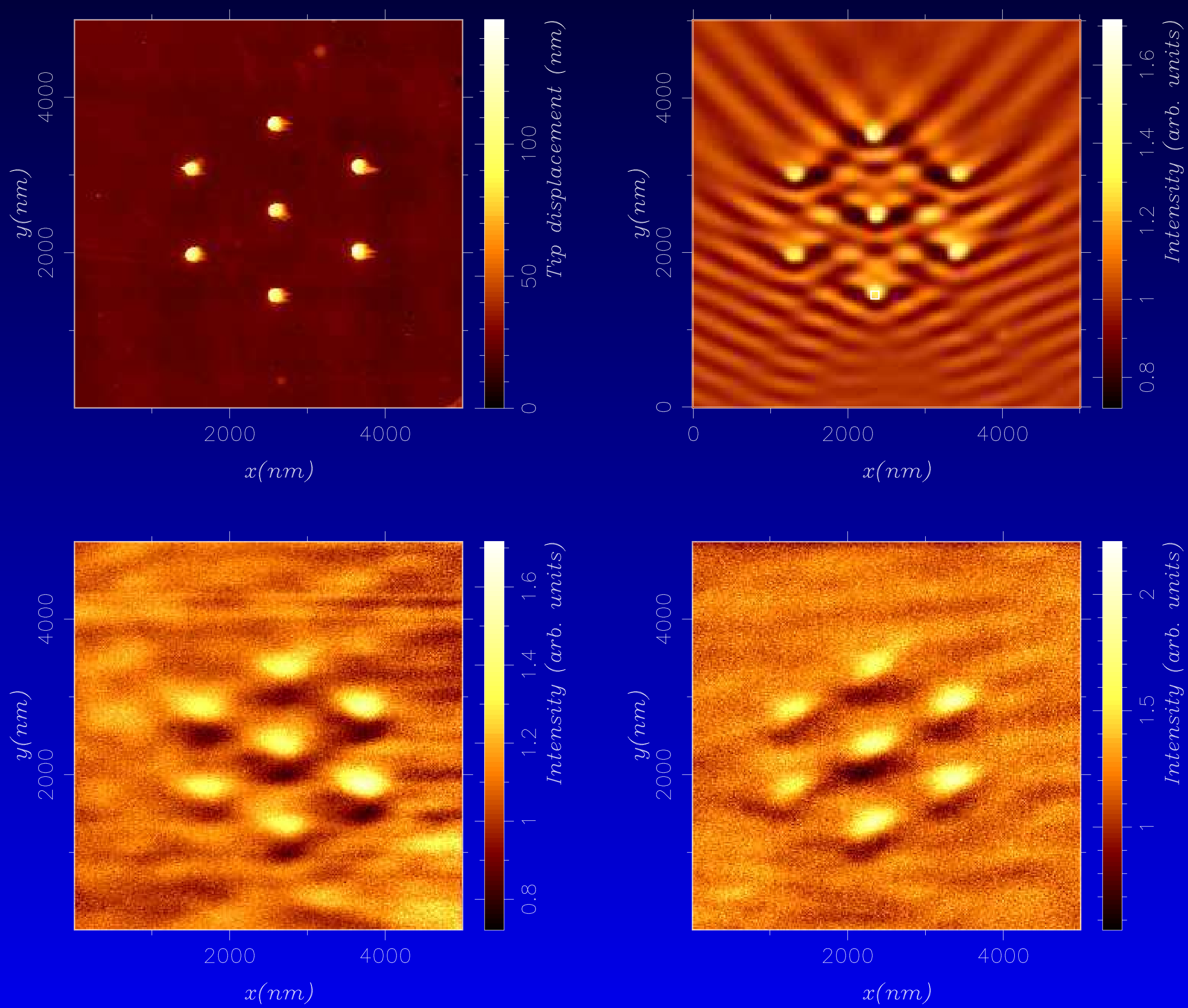
$|\overrightarrow{E}(\overrightarrow{r})|^2$ (theory, no tip)



$\lambda = 633$ nm ; $\theta = 60$ deg

Dielectric tips: λ dependence

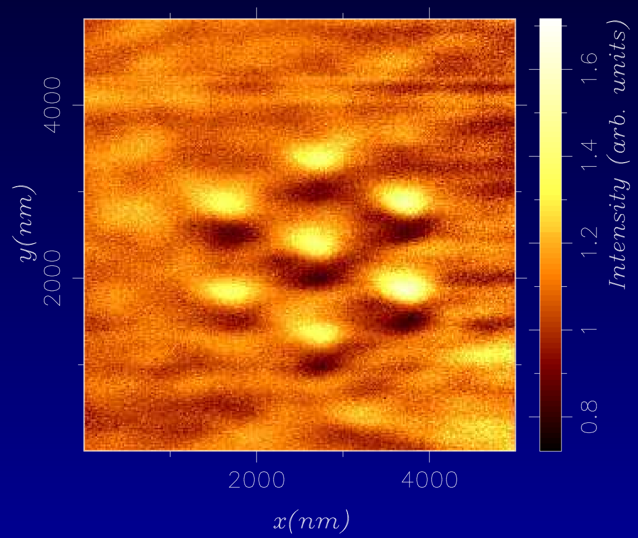
AFM (130x130x100 nm^3) | $|\vec{E}(\vec{r})|^2$ (theory, no tip)



PSTM, λ=633 nm, TM | PSTM, λ=543 nm, TM

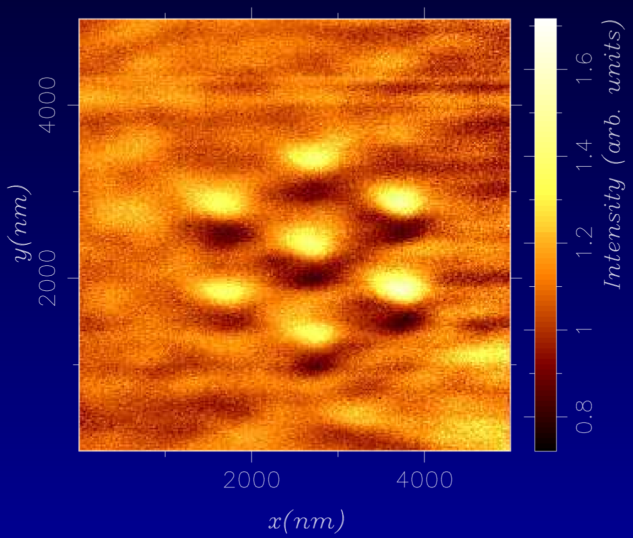
Detecting $|\vec{H}(\vec{r})|^2$ @ $\lambda = 633$ nm

Dielectric tip, TM

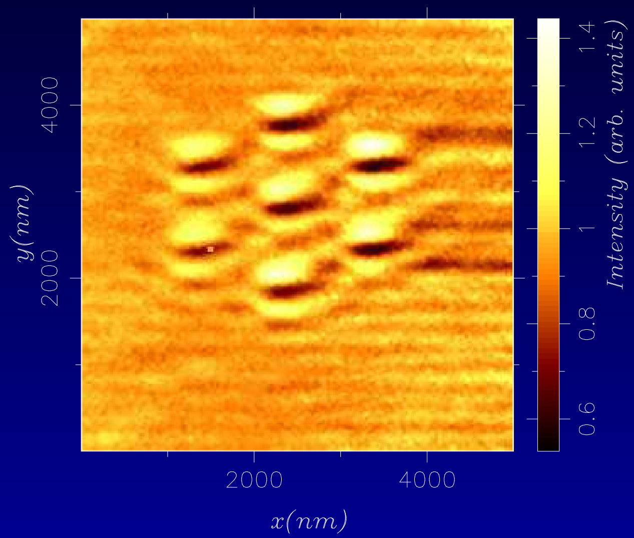


Detecting $|\overrightarrow{H}(\overrightarrow{r})|^2$ @ $\lambda = 633$ nm

Dielectric tip, TM

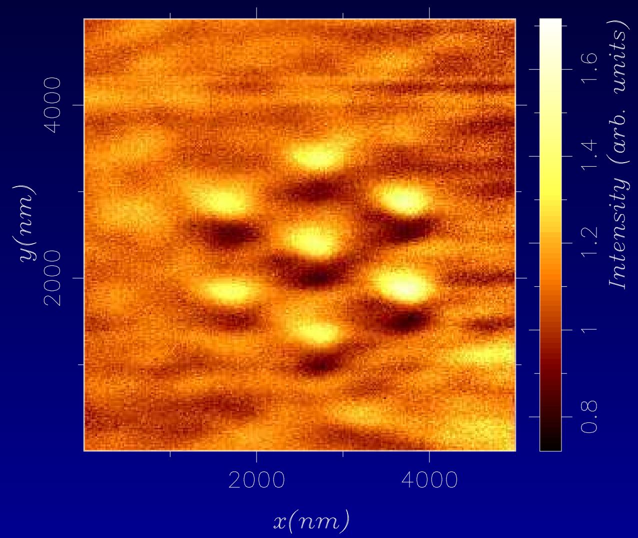


Gold coated tip d =20 nm

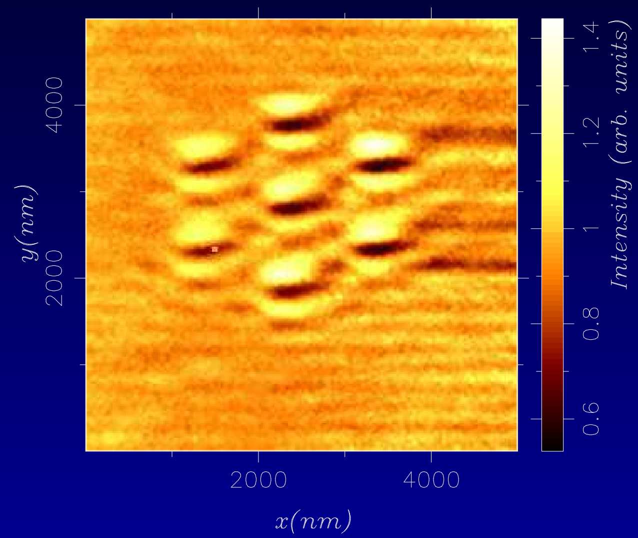


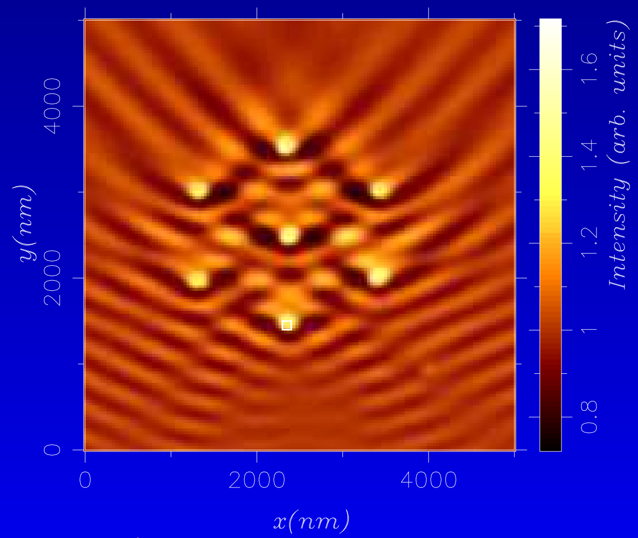
Detecting $|\vec{H}(\vec{r})|^2$ @ $\lambda = 633$ nm

Dielectric tip, TM



Gold coated tip d =20 nm



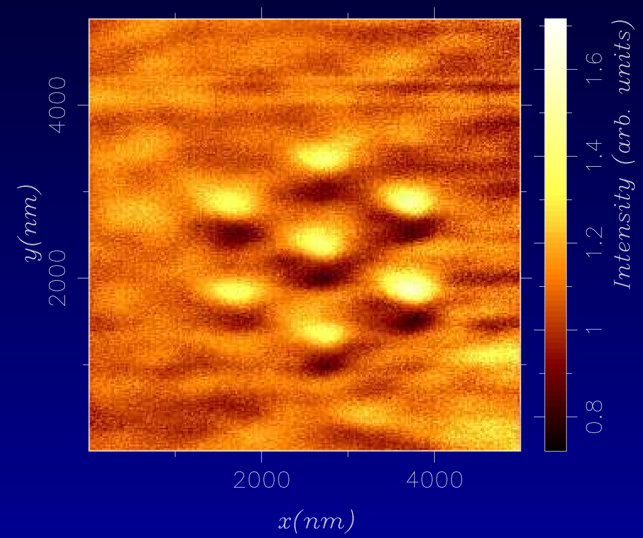


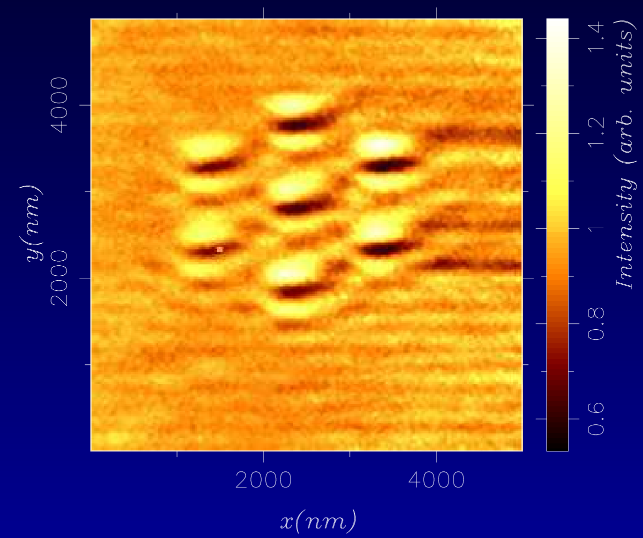
$|\vec{E}(\vec{r})|^2$ (theory, no tip)

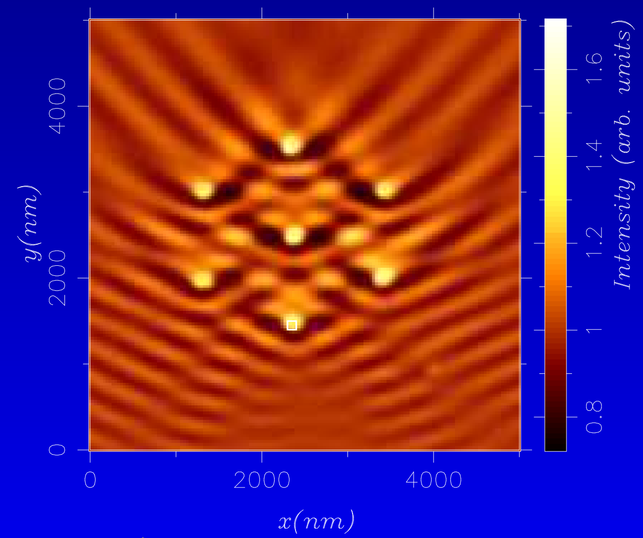
Detecting $|\overrightarrow{H}(\overrightarrow{r})|^2$ @ $\lambda = 633$ nm

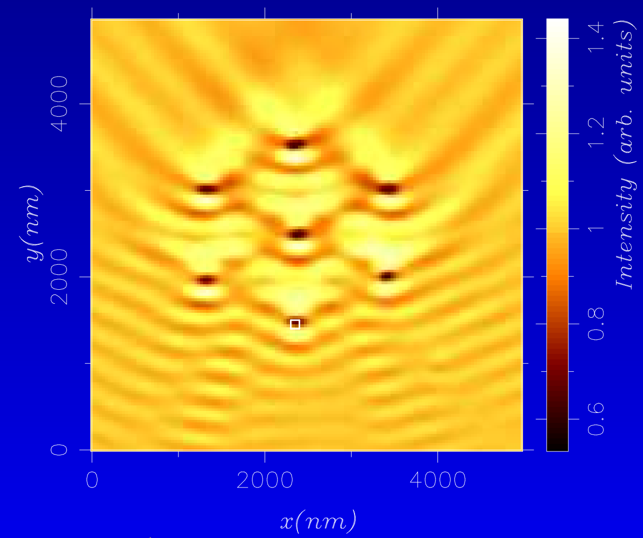
Dielectric tip, TM

Gold coated tip d =20 nm





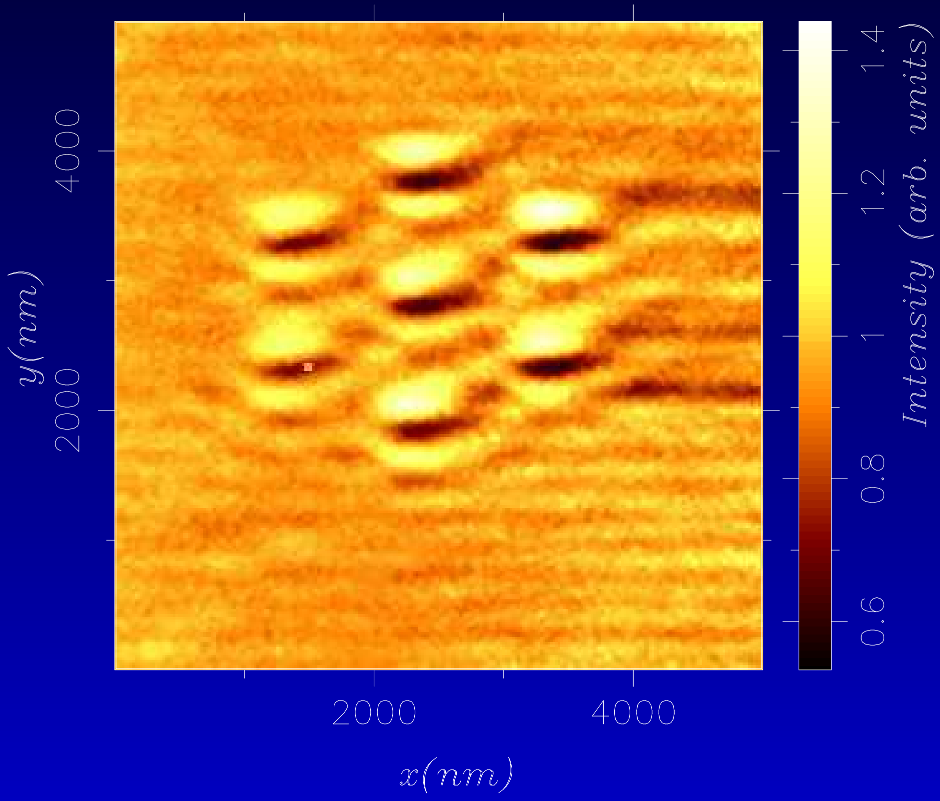


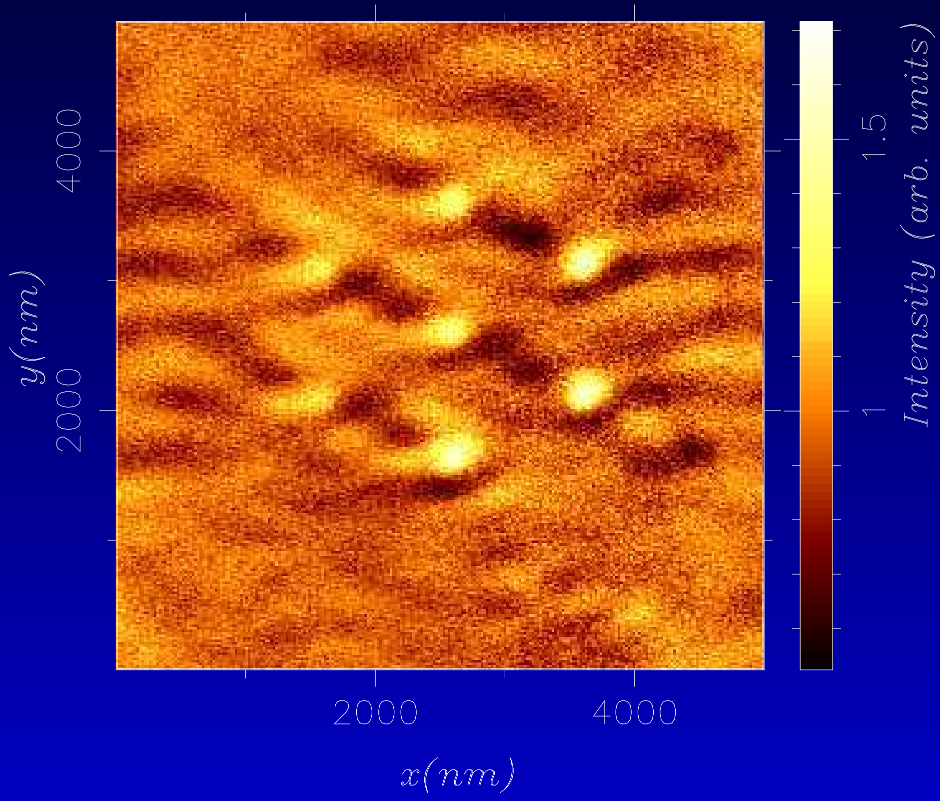


$|\overrightarrow{E}(\overrightarrow{r})|^2$ (theory, no tip)

$|\overrightarrow{H}(\overrightarrow{r})|^2$ (theory, no tip)

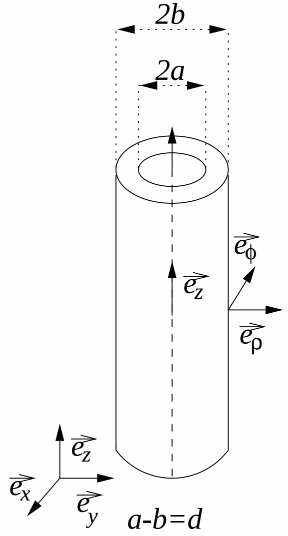
Dependence on circular symmetry

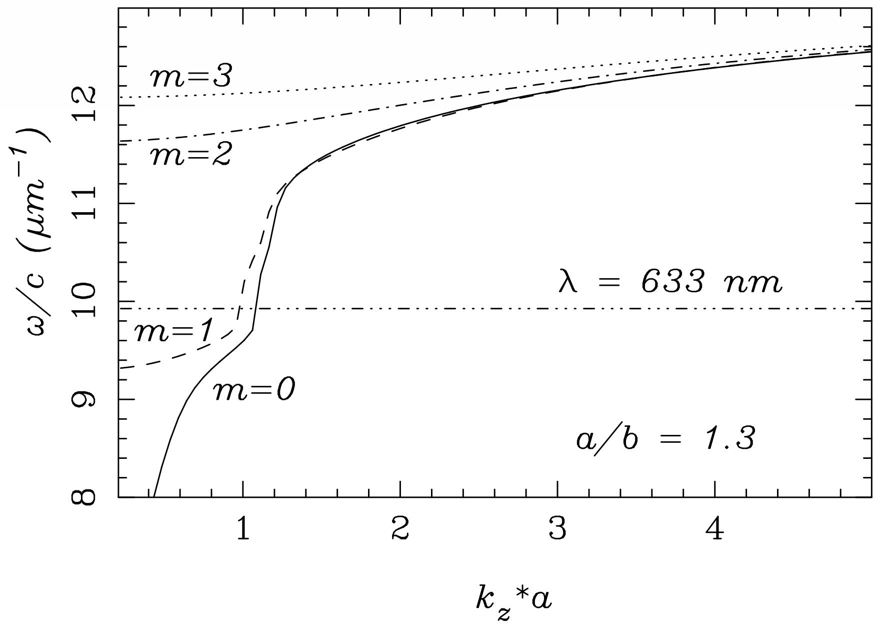


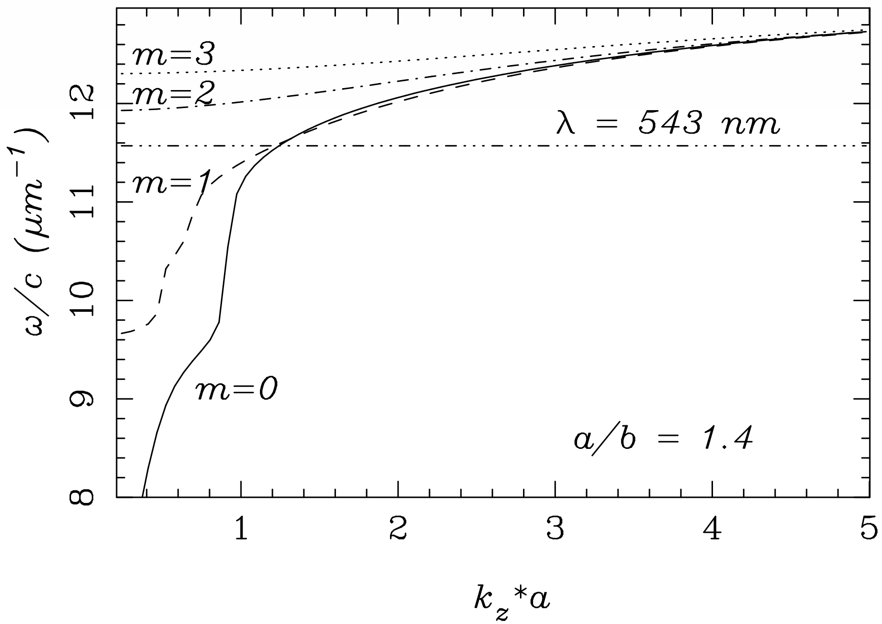


- $\lambda = 633$ nm ; $\theta = 60$ deg ; TM polarization
- E. Devaux, A. Dereux *et al.*, Phys. Rev. B. **62**, 10504 (2000)

Circular symmetry plasmons: non-retarded







E. Devaux, A. Dereux *et al.*, Phys. Rev. B. **62**, 10504 (2000)

Dipole moments of nanostructures

Coupling of nanostructures eigenmodes to an external field

$$\mathbf{p} \cdot \mathbf{E}_{\mathrm{ext}} + \mathbf{m} \cdot \mathbf{B}_{\mathrm{ext}}$$

Electric dipole moment

$$\mathbf{p} = \frac{1}{i\omega} \int d\mathbf{r} \;\; \mathbf{J}(\mathbf{r})$$

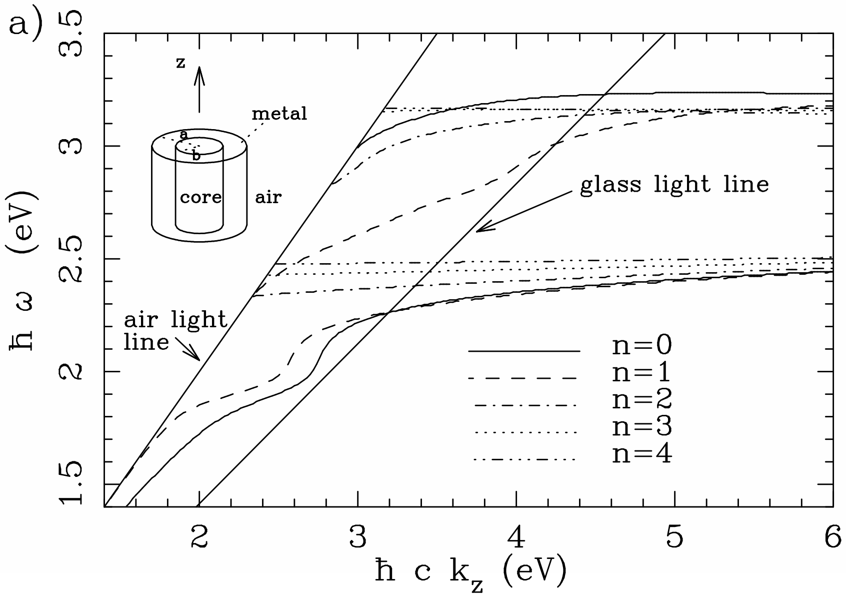
Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2c} \int d\mathbf{r} \;\; [\mathbf{r} \times \mathbf{J}(\mathbf{r})]$$

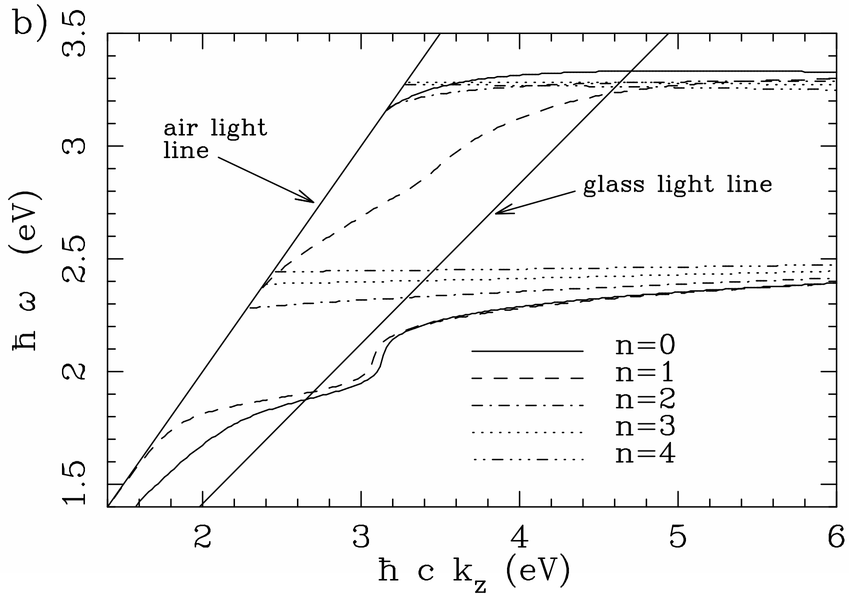
where (Gaussian Units)

$$\mathbf{J}(\mathbf{r}) = \frac{i\omega}{4\pi} \; [\epsilon(\mathbf{r}, \omega) \quad \epsilon_d] \;\; \mathbf{E}(\mathbf{r})$$

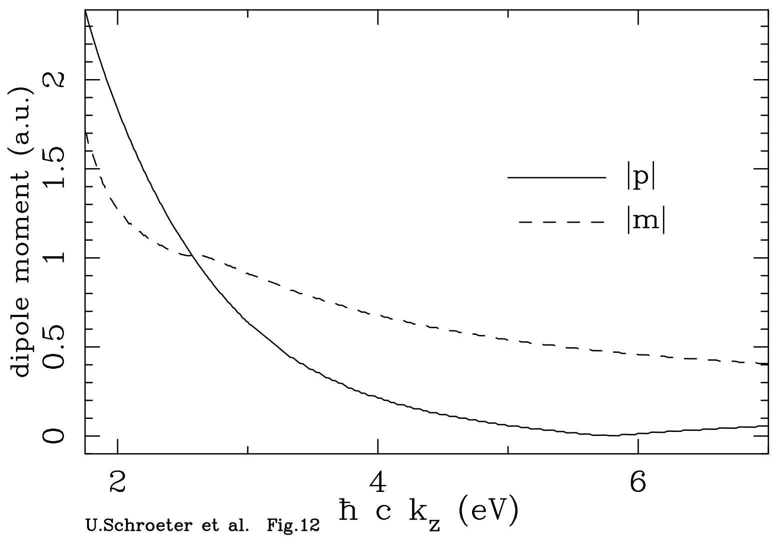
Circular symmetry plasmons: retarded



U.Schroeter et al. Fig.2a



U.Schroeter et al. Fig.2b

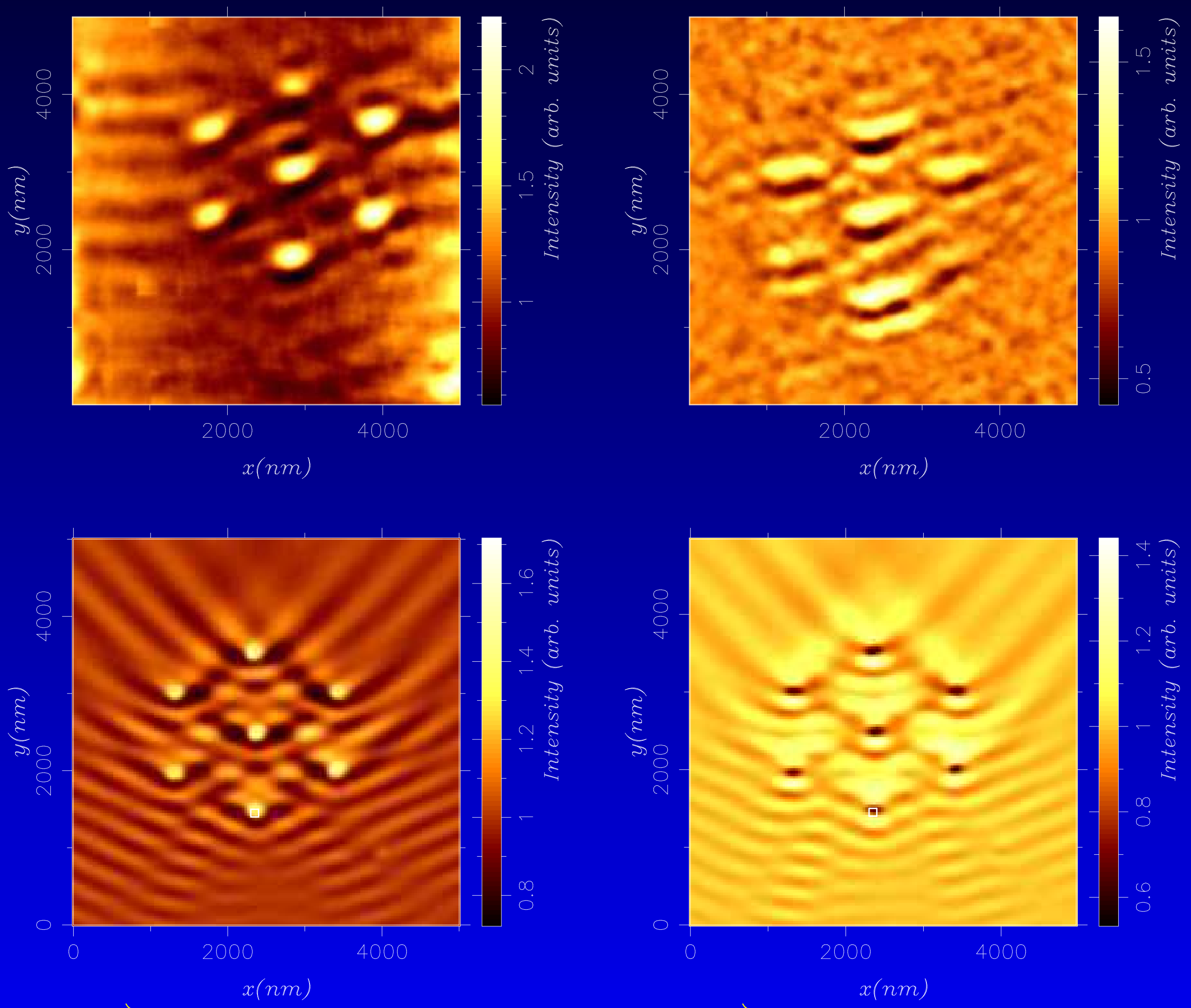


U.Schroeter et al. Fig.12

Detecting $|\overrightarrow{H}(\overrightarrow{r})|^2$ @ $\lambda = 543$ nm

Gold coated tip d =20 nm

Gold coated tip d =30 nm

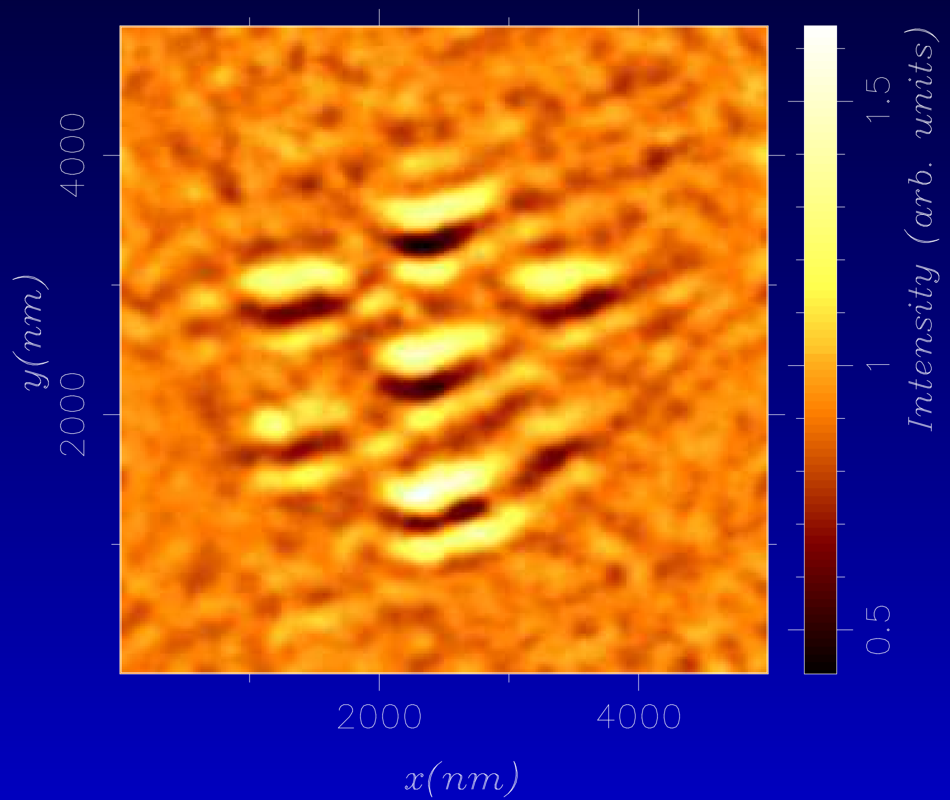


$|\overrightarrow{E}(\overrightarrow{r})|^2$ (theory, no tip)

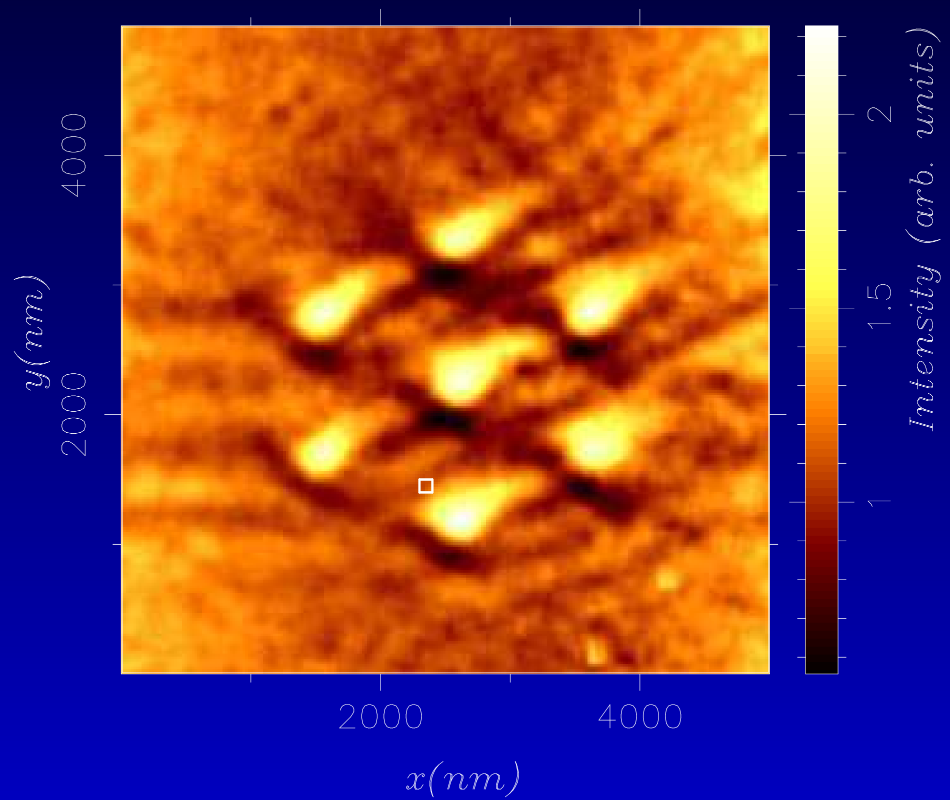
$|\overrightarrow{H}(\overrightarrow{r})|^2$ (theory, no tip)

Dependence on coating thickness

Gold coated tip d =30 nm



Gold coated tip d =35 nm

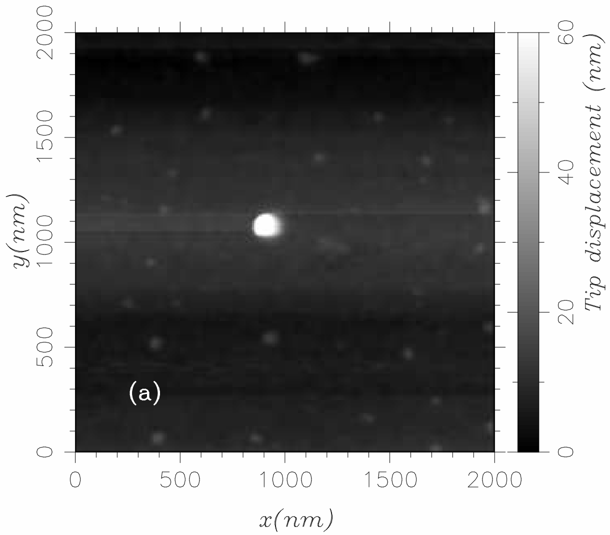


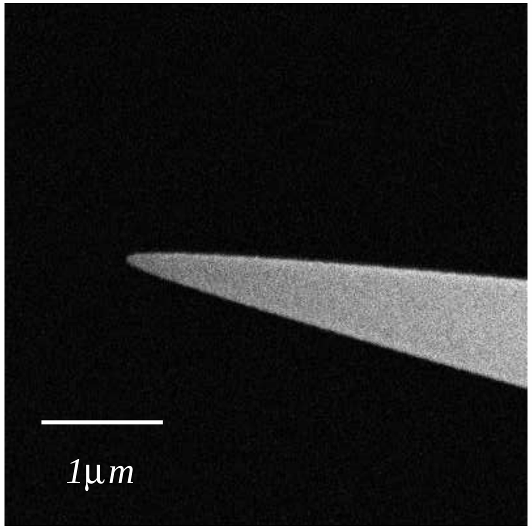
- $\lambda = 543$ nm ; $\theta = 60$ deg ; TM polarization
- E. Devaux, A. Dereux *et al.*, Phys. Rev. B. **62**, 10504 (2000)

Gold particles on glass

AFM ($100x100x60\ nm^3$)

Cr coated tip

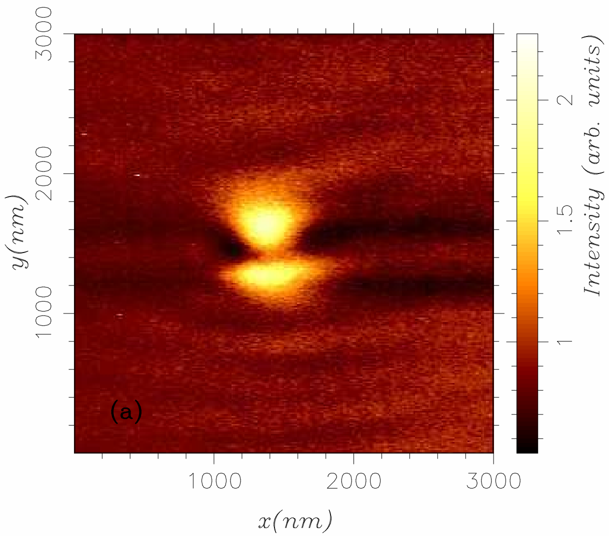




E. Devaux, Ph D Thesis, Univ. Bourgogne, Dijon (2000)

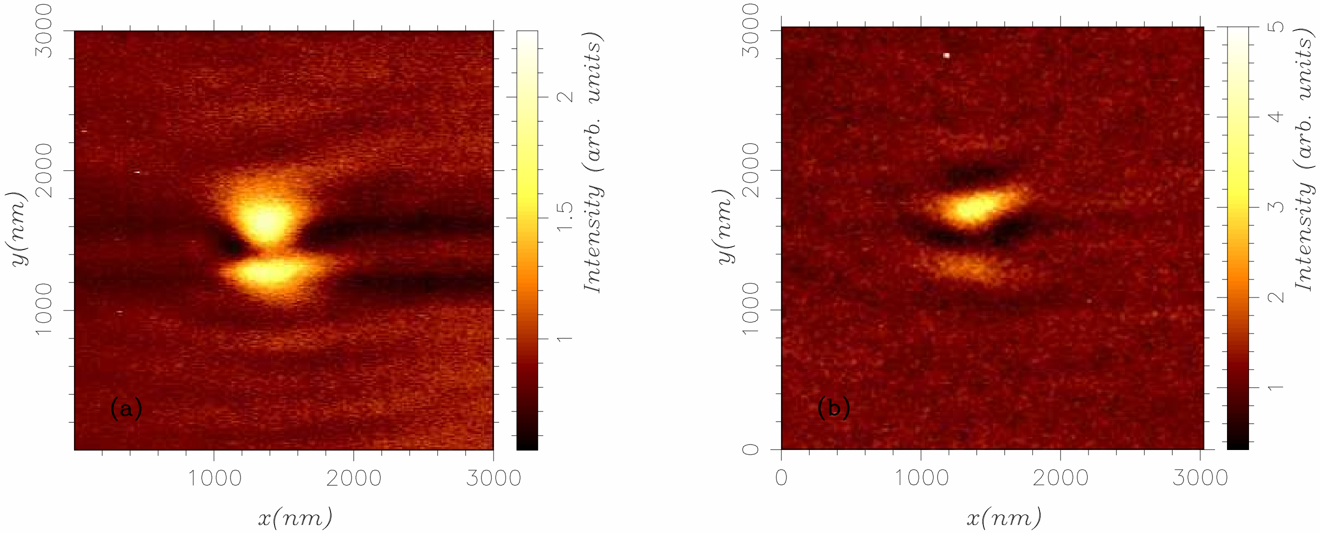
Detecting $|\overrightarrow{H}(\overrightarrow{r})|^2$ @ $\lambda =$ 633 nm

Cr coated tip d =7nm



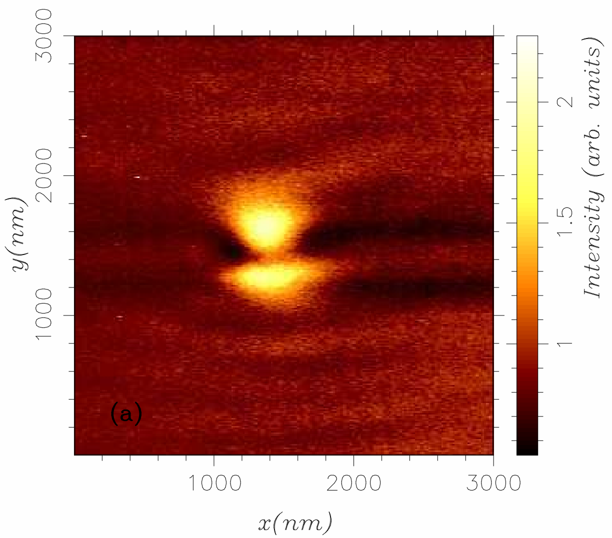
Detecting $|\overrightarrow{H}(\overrightarrow{r})|^2$ @ $\lambda = 633$ nm

Cr coated tip d =7nm — Gold coated tip d =20 nm

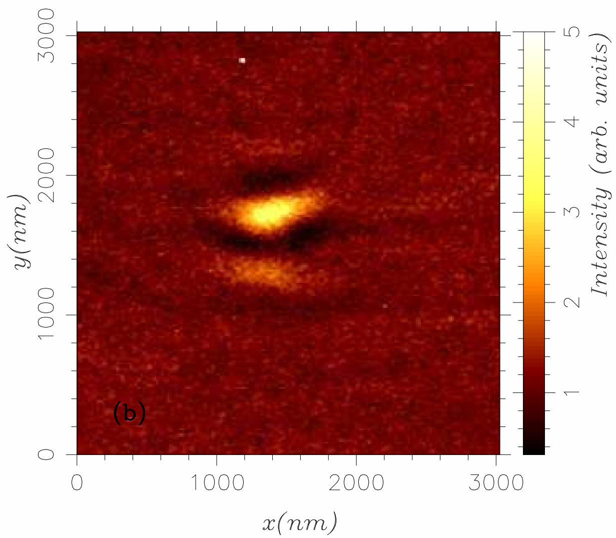


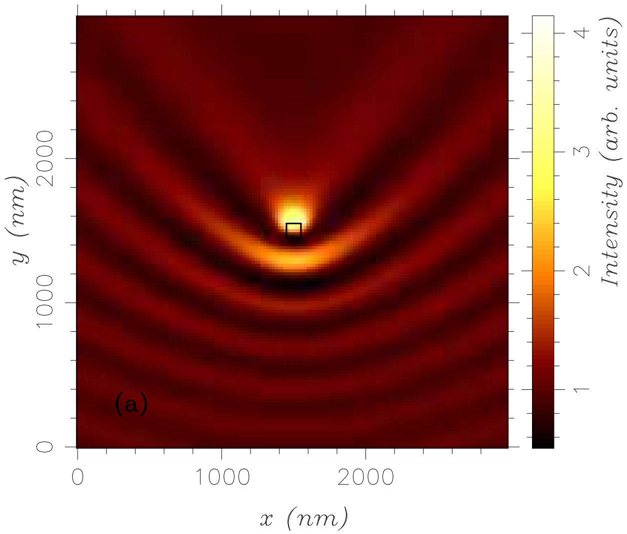
Detecting $|\vec{H}(\vec{r})|^2$ @ $\lambda = 633$ nm

Cr coated tip d =7nm



Gold coated tip d =20 nm

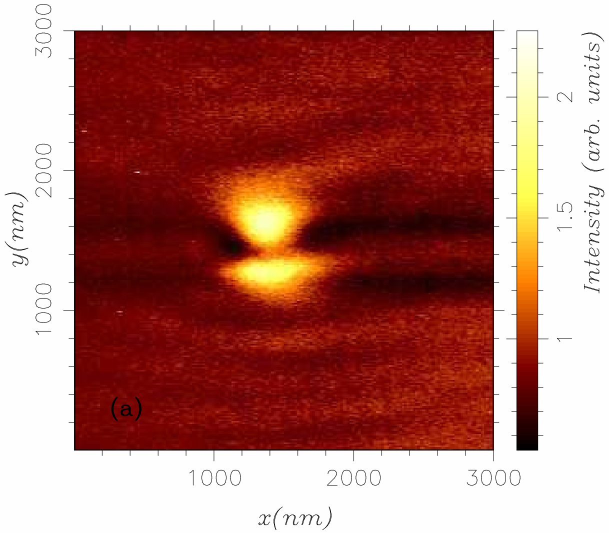


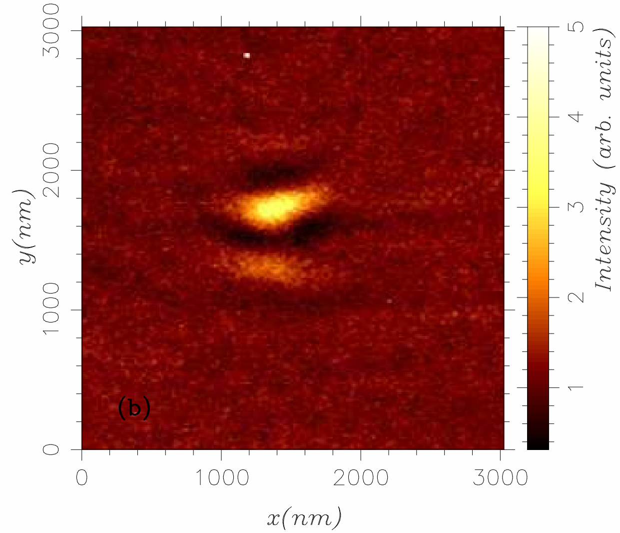


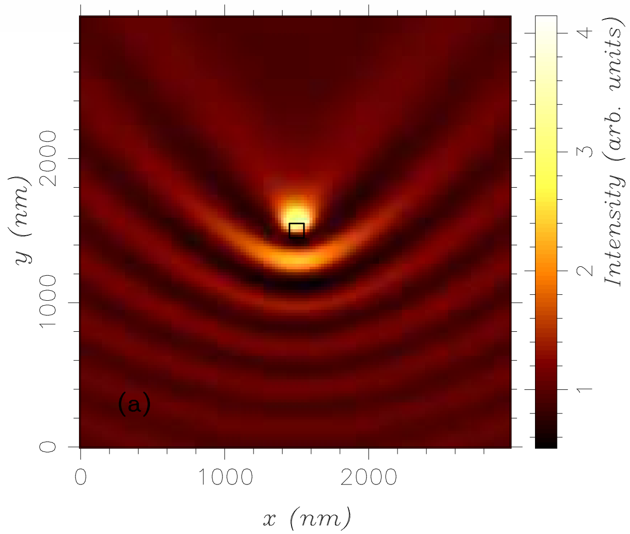
Detecting $|\vec{H}(\vec{r})|^2$ @ $\lambda = 633$ nm

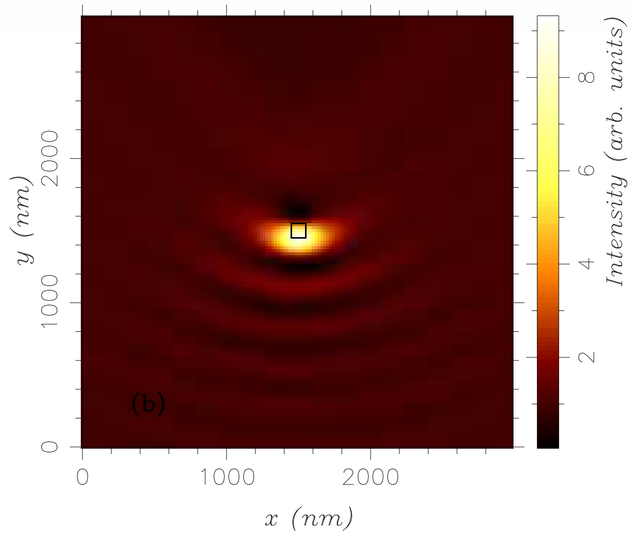
Cr coated tip d =7nm

Gold coated tip d =20 nm





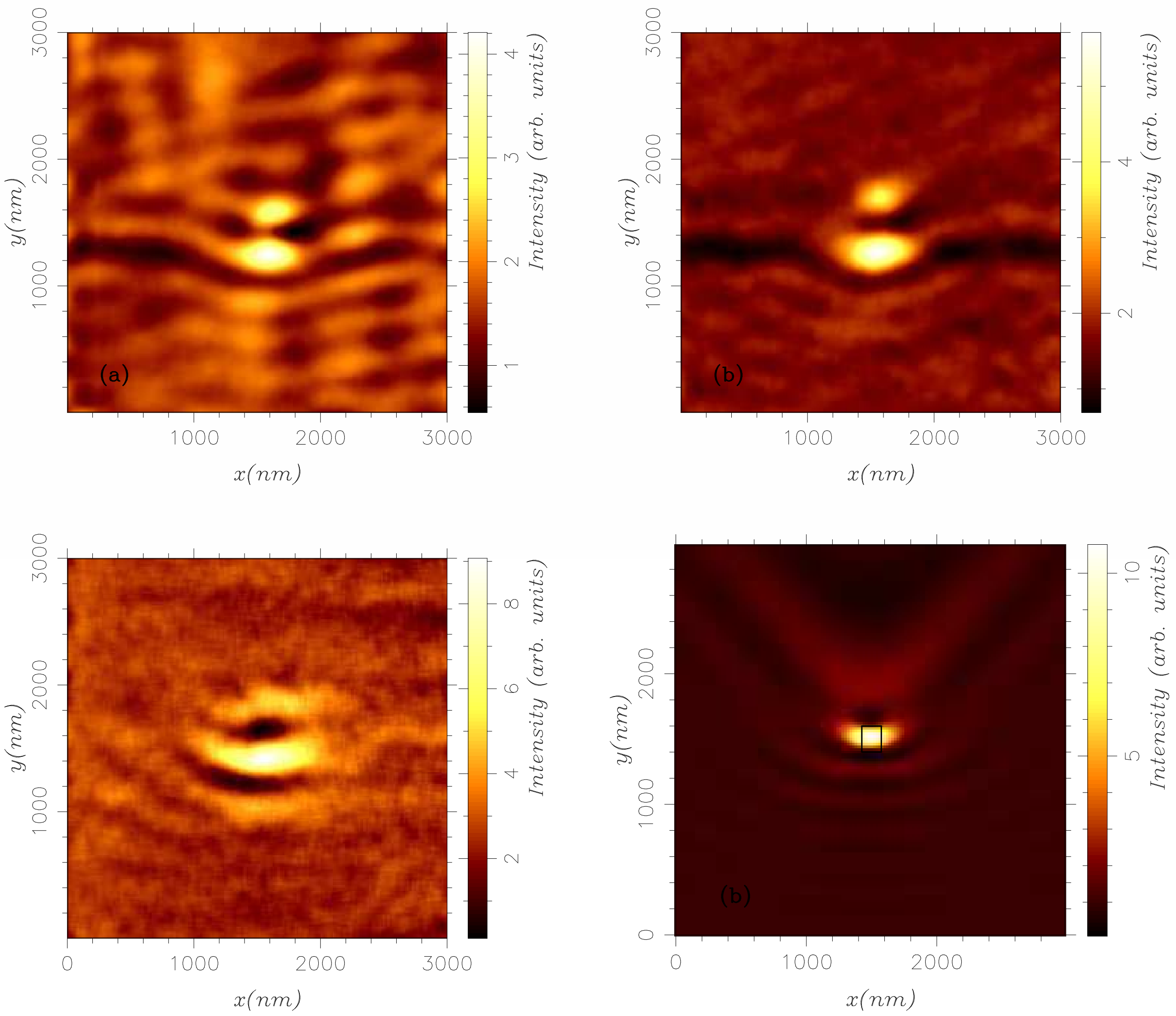




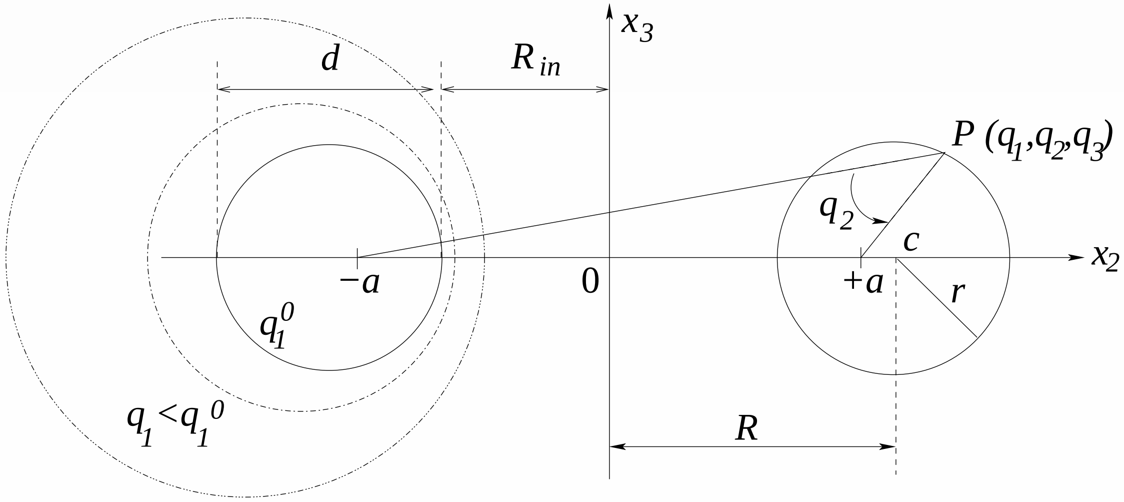
Detecting $|\vec{H}(\vec{r})|^2$ @ $\lambda = 543$ nm

Cr coated tip d =7 nm

Gold coated tip d =30 nm



Torus Surface Plasmons



Non-retarded approximation

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = 0$$

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0$$

$$\nabla^2 \Phi(\mathbf{r}) = 0$$

$$\Phi(\mathbf{r}) = \sqrt{f(q_1, q_2)}\ L(q_1)\ U(q_2)\ V(q_3)$$

A. Mary, A. Dereux, T. L. Ferrell, to be published

General solutions in toroidal coord.

$$\Phi(\mathbf{r}) = \sqrt{f(q_1, q_2)}\; L(q_1)\; U(q_2)\; V(q_3)$$

$$f(q_1, q_2) = \cosh q_1 \quad \cos q_2$$

$$U_n(q_2) = \begin{cases} \cos n q_2 & (\text{even}) \\ \\ \sin n q_2 & (\text{odd}) \end{cases}$$

$$V_m(q_3) = \mathrm{e}^{imq_3}$$

$$\frac{1}{\sinh q_1} \frac{\partial}{\partial q_1} \left(\sinh q_1 \frac{\partial L^m_{n \quad \frac{1}{2}}(q_1)}{\partial q_1} \right) \quad \left[\frac{m^2}{\sinh^2 q_1} + \left(n^2 \quad \frac{1}{4} \right) \right] L^m_{n \quad \frac{1}{2}}(q_1$$

General form of solutions

$$L^m_{n\ \frac{1}{2}}(q_1) = \begin{cases} P^m_{n\ \frac{1}{2}}(\cosh q_1) \\ \\ Q^m_{n\ \frac{1}{2}}(\cosh q_1) \end{cases}$$

$$\lim_{q_1 \to \infty} P^m_{n\ \frac{1}{2}}(\cosh q_1) = \infty$$

$$\lim_{q_1 \to 0} Q^m_{n\ \frac{1}{2}}(\cosh q_1) = \infty$$

$$\Phi(\mathbf{r}) = \begin{cases} \Phi^{\text{in}}(\mathbf{r}) & \text{if } q_1 > q_1^0 \\ \\ \Phi^{\text{out}}(\mathbf{r}) & \text{if } q_1 < q_1^0 \end{cases}$$

where

$$\Phi^{\text{in}}(\mathbf{r}) = \sqrt{f(q_1, q_2)} \sum_{m=\infty}^{+\infty} \sum_{n=0}^{+\infty} B^m_n Q^m_{n\ \frac{1}{2}} U_n(q_2)\ e^{imq_3}$$

$$\Phi^{\text{out}}(\mathbf{r}) = \sqrt{f(q_1, q_2)} \sum_{m=\infty}^{+\infty} \sum_{n=0}^{+\infty} A^m_n P^m_{n\ \frac{1}{2}} U_n(q_2)\ e^{imq_3}$$

Boundary conditions on torus surface

Continuity of the tangential components of the electric field $\mathbf{E}(\mathbf{r})$

$$B_n^m Q_{n\ \frac{1}{2}}^{m\ 0} = A_n^m P_{n\ \frac{1}{2}}^{m\ 0}$$

continuity of the normal component of $\mathbf{D}(\mathbf{r}) = \epsilon(\mathbf{r},\omega)\ \mathbf{E}(\mathbf{r})$

$$\epsilon(\omega)\ \frac{\partial \Phi^{\mathrm{in}}(\mathbf{r})}{\partial q_1}\Bigg]_{q_1=q_1^0} = \epsilon_d\ \frac{\partial \Phi^{\mathrm{out}}(\mathbf{r})}{\partial q_1}\Bigg]_{q_1=q_1^0}$$

leads to

$$\sum_{n=0}^{\infty} A_n^M\ U_n(q_2)\ C_n^M = 2\sum_{n=0}^{\infty} A_n^M\ U_n(q_2)\ G_n^M\ \ \cos q_2$$

where

$$G_n^m = \epsilon_d Q_{n\ \frac{1}{2}}^{m\ 0}\ \frac{dP_{n\ \frac{1}{2}}^{m}}{dq_1}\Bigg]_{q_1=q_1^0} \qquad \epsilon(\omega) P_{n\ \frac{1}{2}}^{m\ 0}\ \frac{dQ_{n\ \frac{1}{2}}^{m}}{dq_1}\Bigg]_{q_1=q_1^0}$$

and

$$C_n^M = \Big[(\epsilon(\omega) \quad \epsilon_d) \sinh q_1^0 P_{n\ \frac{1}{2}}^{M\ 0} Q_{n\ \frac{1}{2}}^{M\ 0} \quad 2G_n^M \cosh q_1^0\Big]$$

Eigenmodes

Condition for even modes where $N \geq 0$.

$$C_N^M \; C_{N+1}^M = (1 + \delta_{0,N}) \; G_N^M \; G_{N+1}^M$$

Condition for odd modes where $N > 0$.

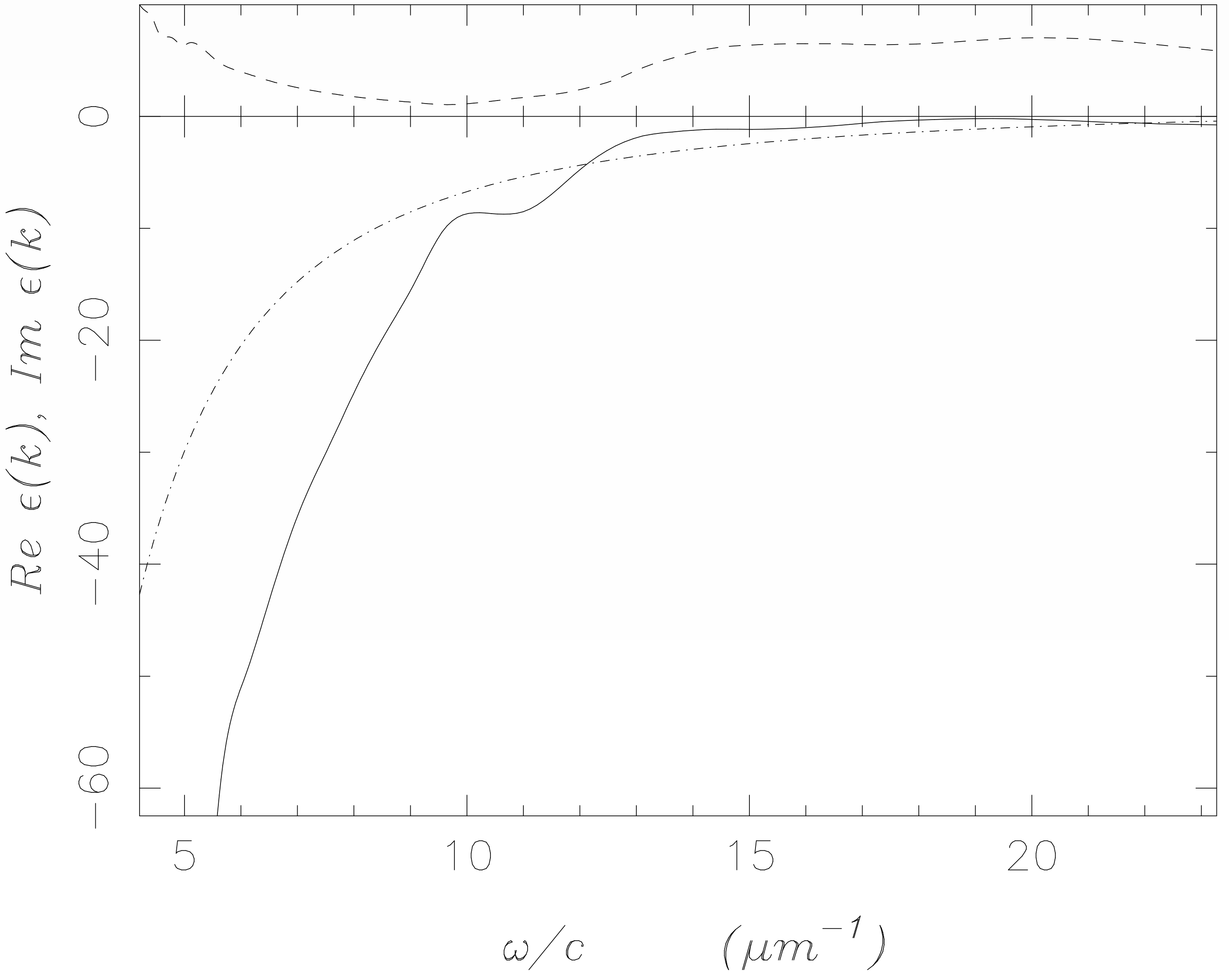
$$C_N^M \; C_{N+1}^M = G_N^M \; G_{N+1}^M$$

We label a mode only with the couple (N, M), thereby meaning the triplet $(N, N+1, M)$

$$\Phi_{N,M}^{\text{in}}(\mathbf{r}) = \sqrt{f(q_1, q_2)} \sum_{n=N}^{N+1} A_n^M Q_{n\;\;\frac{1}{2}}^M \; U_n(q_2) \; e^{iMq_3}$$

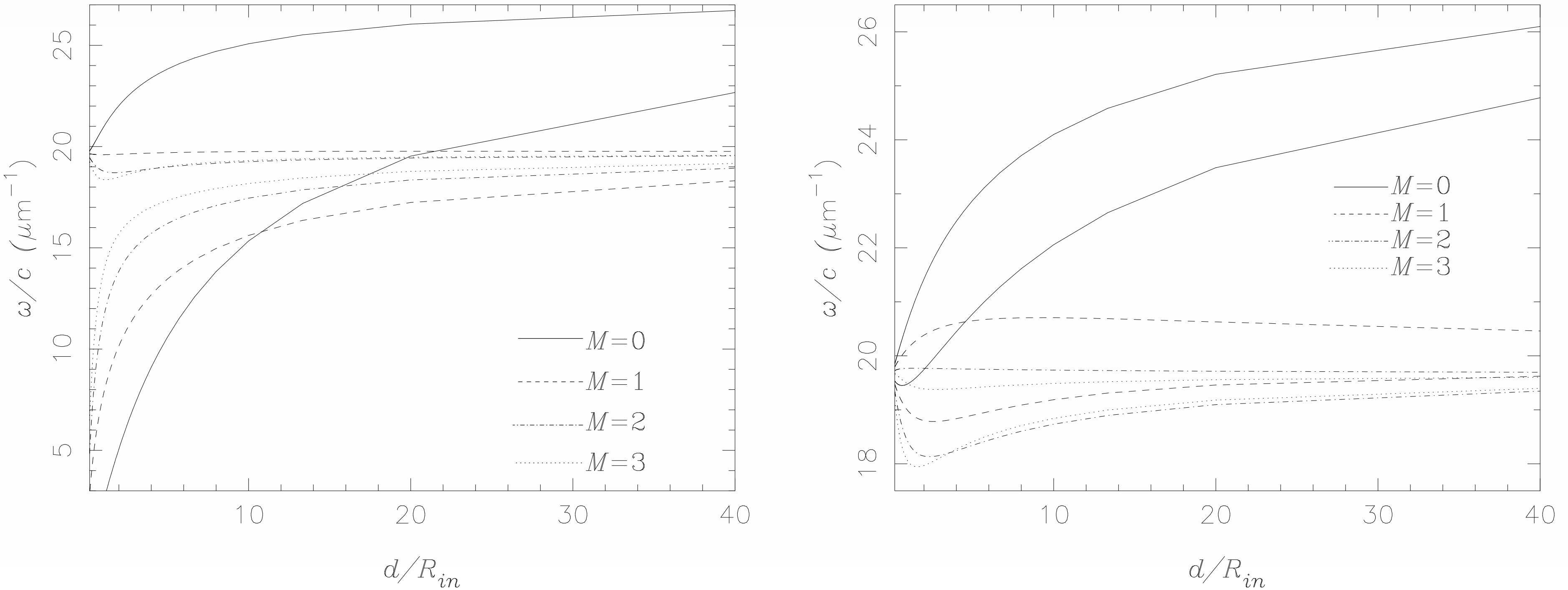
$$\Phi_{N,M}^{\text{out}}(\mathbf{r}) = \sqrt{f(q_1, q_2)} \sum_{n=N}^{N+1} A_n^M P_{n\;\;\frac{1}{2}}^M \; U_n(q_2) \; e^{iMq_3}$$

Drude vs Exp. Diel. Functions



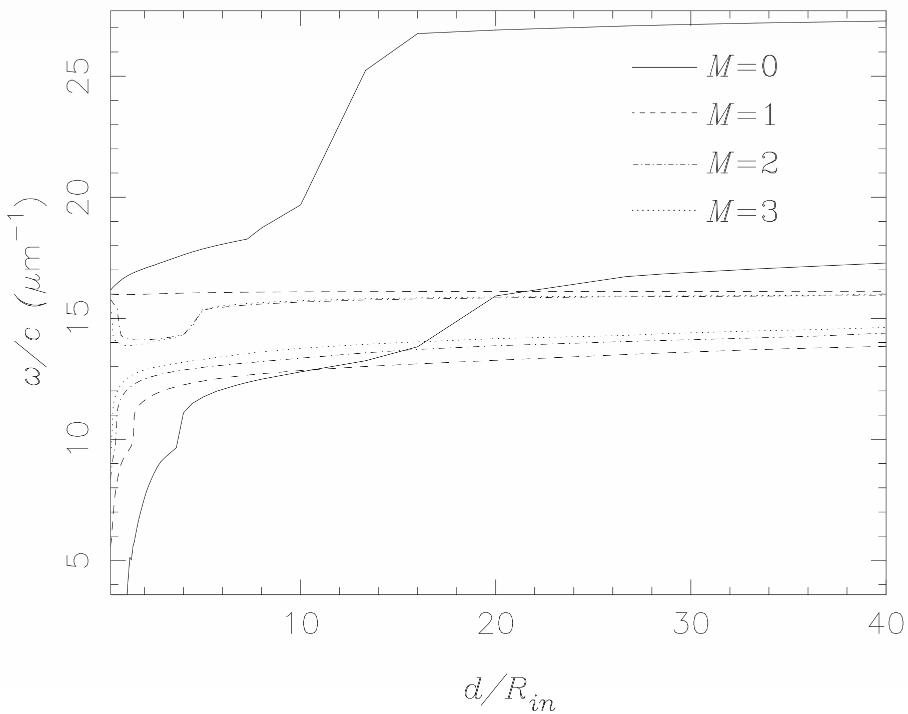
$$\epsilon(\omega) = 1 \quad \frac{\omega_p^2}{\omega^2 + i\gamma(\omega)}$$

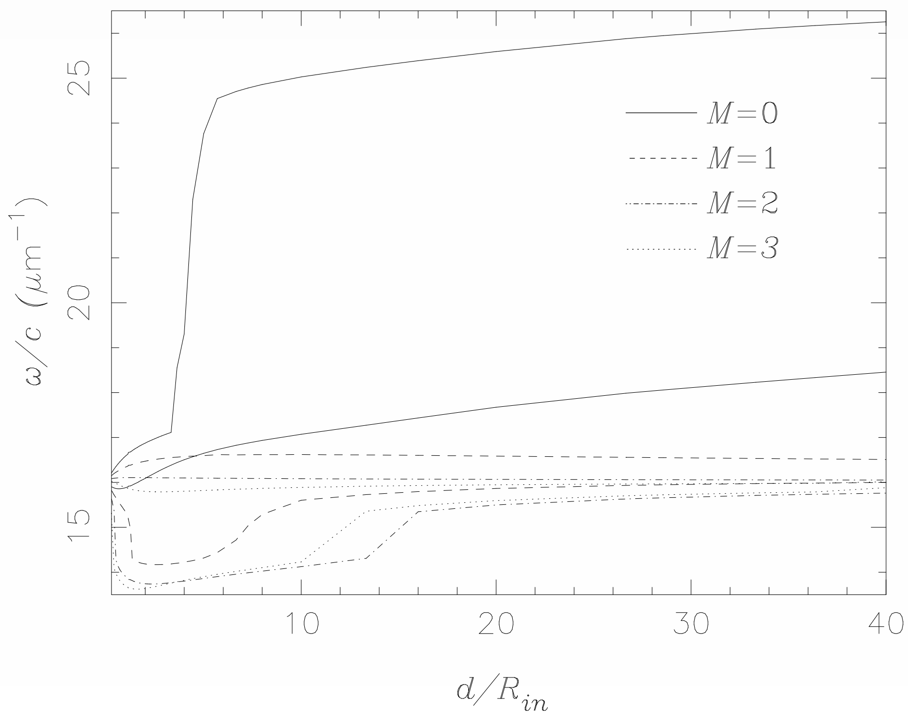
Dispersion relations (Drude diel. fct.)



Au (Drude) torus embedded in vacuum:
$N = 0$, $M = 1, 2, 3$ (left, even modes only) and $N = 1$, $M = 1, 2, 3$ (right)

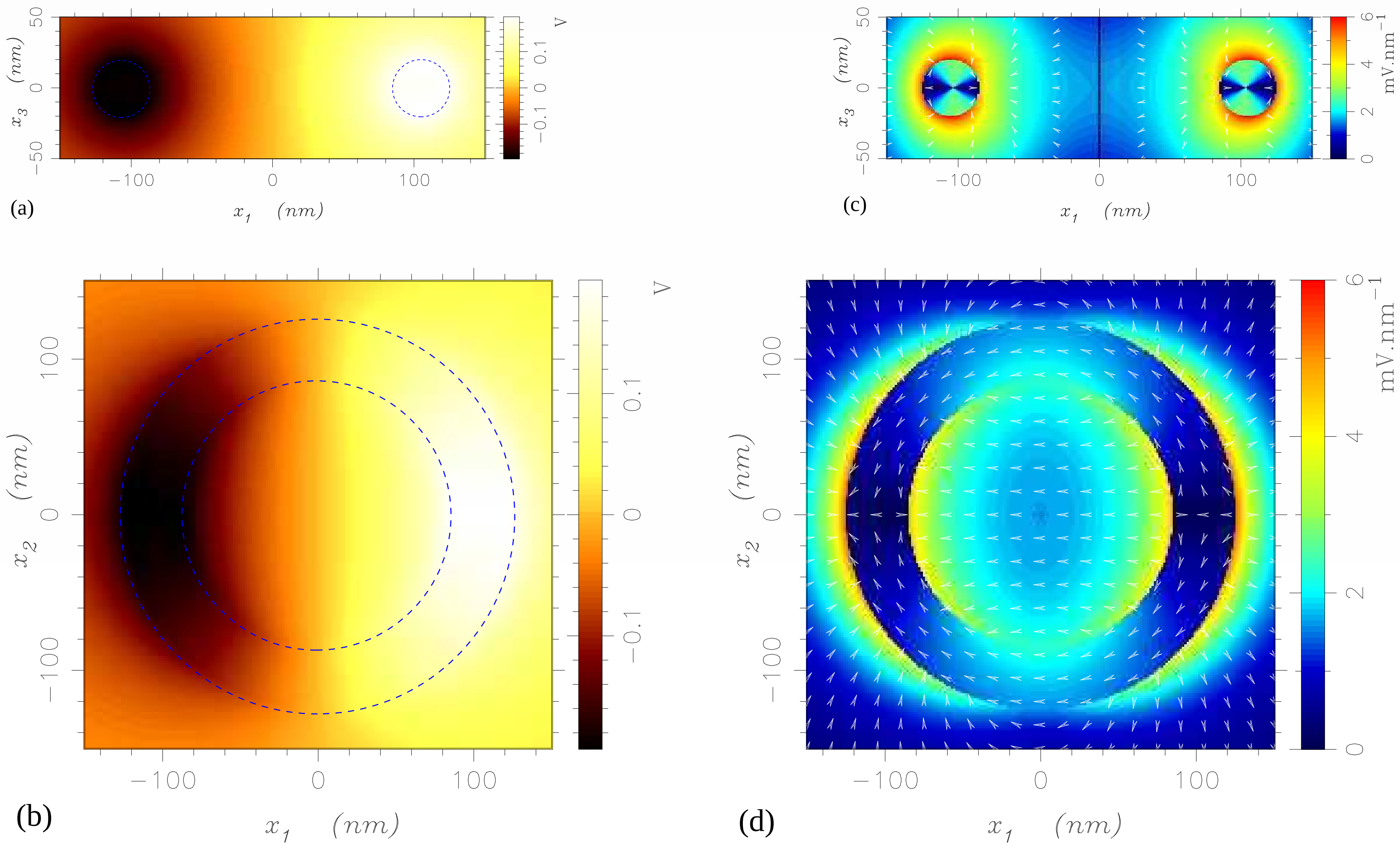
Dispersion relations (Exp. diel. fct)





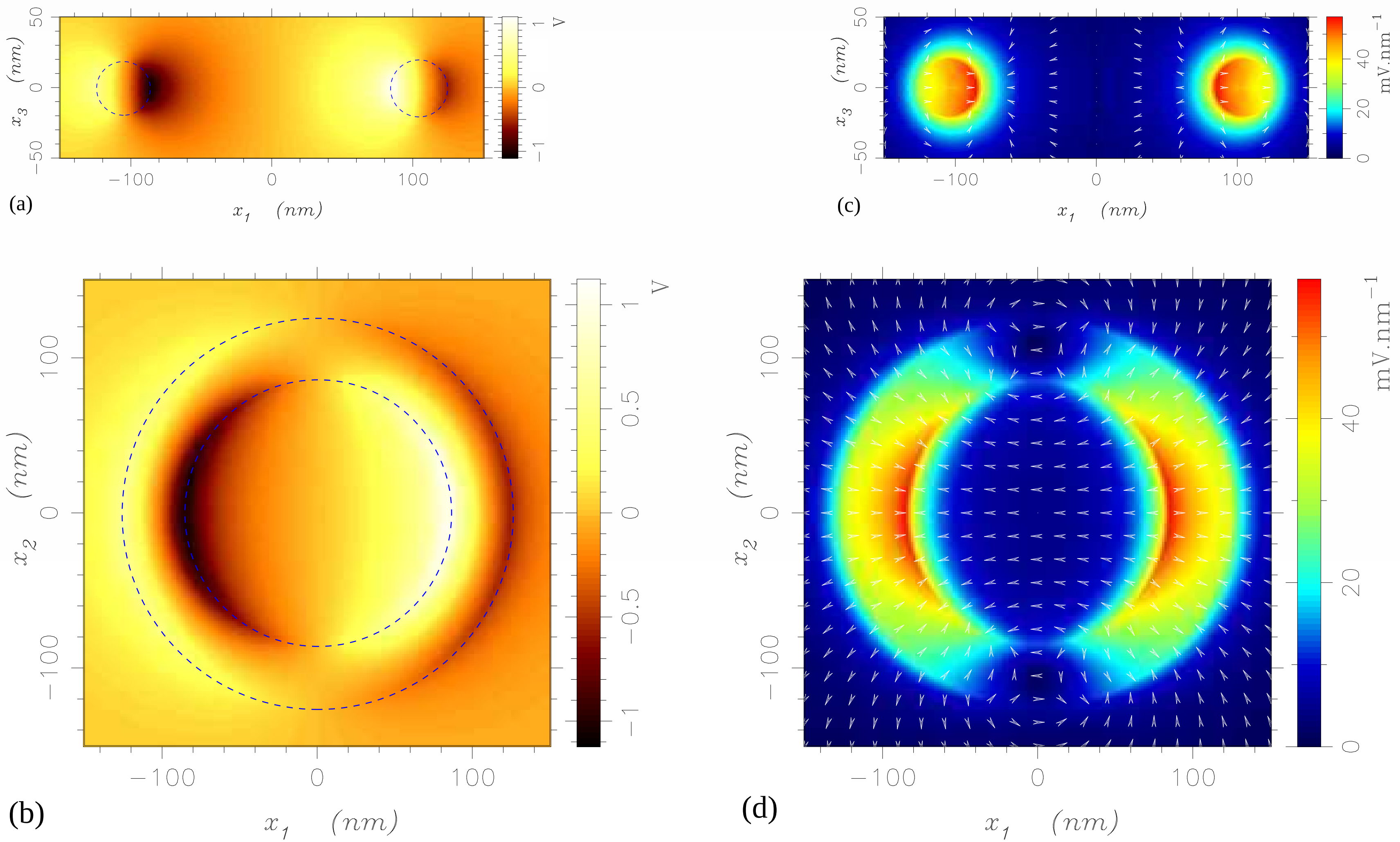
Au (Palik) torus embedded in vacuum:
$N = 0$, $M = 1, 2, 3$ (left, even modes only) and $N = 1$, $M = 1, 2, 3$ (right)

Electric potential & field distributions



For $d = 40$ nm and $R_{in} = 85$ nm, distribution of $\Phi_{0,1}(\mathbf{r})|$ (a,b) and of $|\mathbf{E}_{0,1}(\mathbf{r})|$ (c,d) of the lower frequency ($\omega \ /c = 7.546\ \mu\text{m}\ ^{1}$) mode. (a,c): cut plane $q_3 = 0$. (b,d): cut plane $x_3 = 0$.

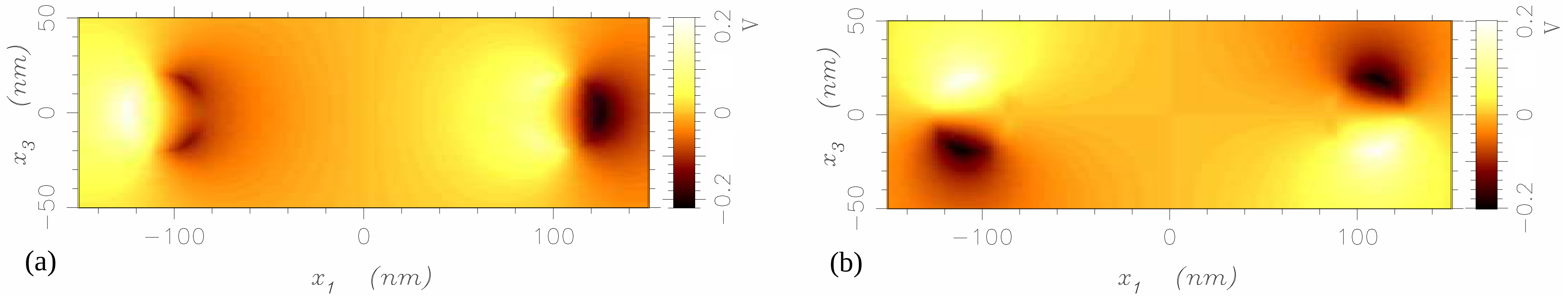
Electric potential & field distributions



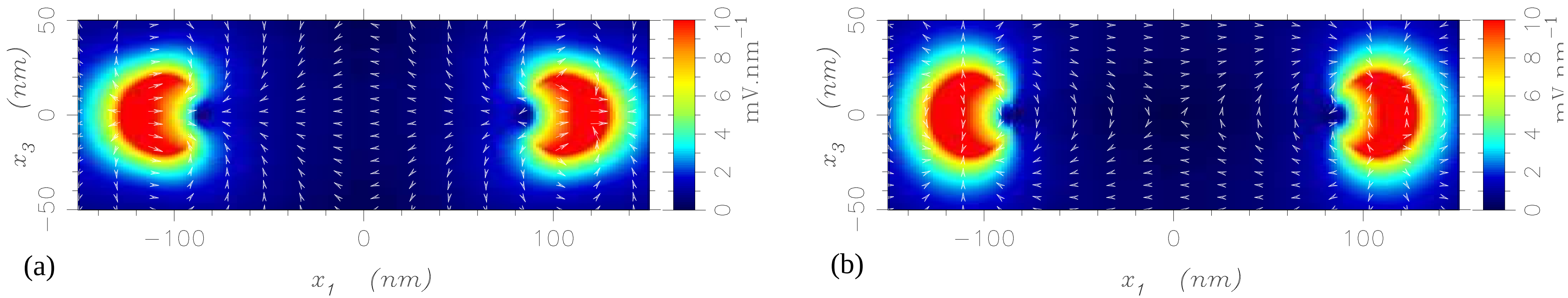
For $d = 40$ nm and $R_{in} = 85$ nm, distribution of $\Phi_{0,1}(\mathbf{r})|$ (a,b) and of $|\mathbf{E}_{0,1}(\mathbf{r})|$ (c,d) of the lower frequency ($\omega \quad /c = 15.99\ \mu\mathrm{m} \quad ^{1}$) mode. (a,c): cut plane $q_3 = 0$. (b,d): cut plane $x_3 = 0$.

Electric potential & field distributions

$d = 40$ nm and $R_{in} = 85$ nm,
higher frequency ($\omega\ /c = 15.67\ \mu\mathrm{m}^{-1}$) mode,
cut plane $q_3 = 0$



Distribution of $\Phi_{1,1}(\mathbf{r})$ (a) Even mode, (b) odd mode.



Distribution of $|\mathbf{E}_{1,1}(\mathbf{r})|$ (a) Even mode, (b) odd mode.

Dipole moments of nanostructures

Coupling of subwavelength torus eigenmodes to an external field

$$\mathbf{p} \cdot \mathbf{E}_{\text{ext}} + \mathbf{m} \cdot \mathbf{B}_{\text{ext}}$$

Electric dipole moment

$$\mathbf{p} = \quad \frac{1}{i\omega} \int d\mathbf{r} \;\; \mathbf{J}(\mathbf{r})$$

Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2c} \int d\mathbf{r} \;\; [\mathbf{r} \times \mathbf{J}(\mathbf{r})]$$

where (Gaussian Units)

$$\mathbf{J}(\mathbf{r}) = \frac{i\omega}{4\pi} \; [\epsilon(\mathbf{r}, \omega) \qquad \epsilon_d] \;\; \mathbf{E}(\mathbf{r})$$

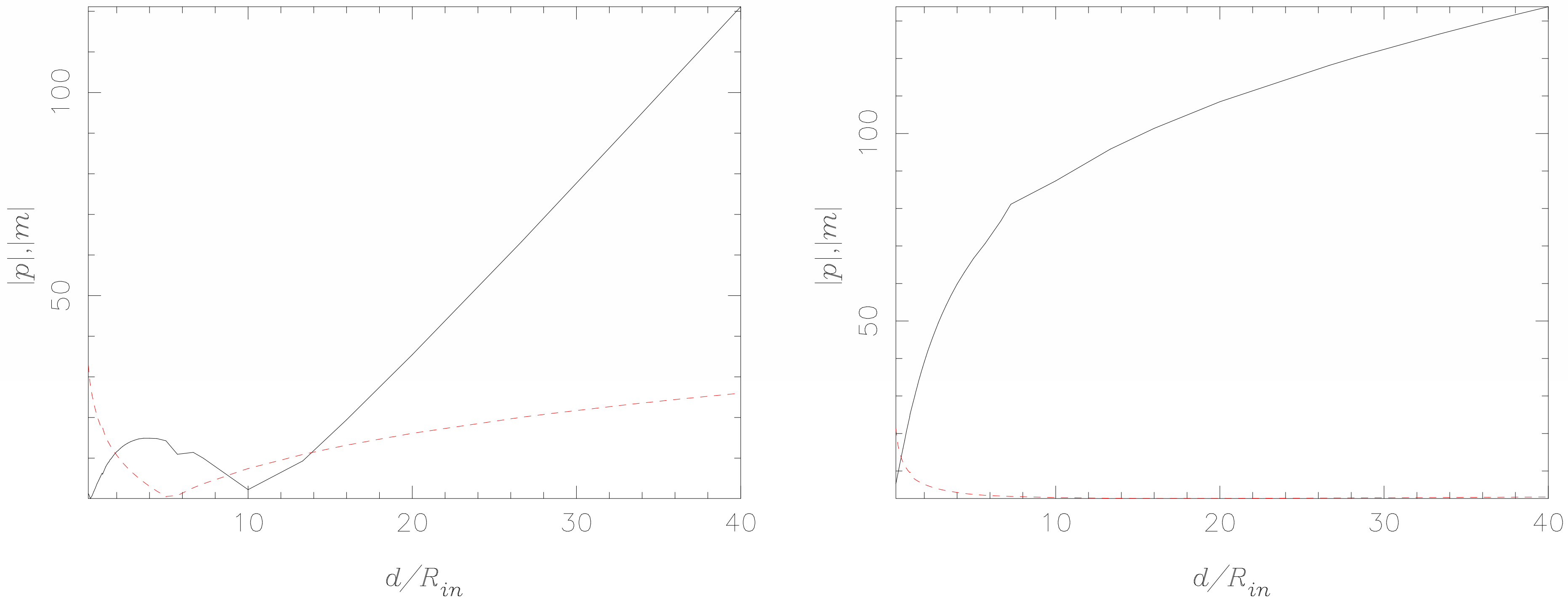
Dipole moments of even and odd modes

For $\cosh(q_1) > 1$ which arises for "well–proportionned" torus ($d \sim R_{in}$), both integrals over q_3 and q_2 may be performed analytically.

$$\mathbf{p}_{N,M} = \begin{cases} \quad \pi(\mathbf{e}_1 + i\mathbf{e}_2)\ \delta_{M,1}\ \int_{q_1^0}^{+\infty} S_1(q_1) dq_1 & \text{(even)} \\ 0 & \text{(odd)} \end{cases}$$

$$\mathbf{m}_{N,M} = \begin{cases} 0 & \text{(even)} \\ \quad \frac{ia\pi\omega}{2c}(i\mathbf{e}_1 \quad \mathbf{e}_2)\delta_{M,1} \int_{q_1^0}^{+\infty} S_2(q_1) dq_1 & \text{(odd)} \end{cases}$$

Dipole moments of nanotorus



Conclusion

Magnetic dipole moments at optical frequencies with odd torus modes

But : even-odd degeneracy !

Remedy : design arrays where unit cell is a couple of torus close to each other !

NoE Plasmo-Nano-Devices workpackage currently running on this issue!